EUI WORKING PAPERS IN ECONOMICS

EUI Working Paper ECO No. 94/34

Estimation Error and the Specification of Unobserved Component Models

AGUSTÍN MARAVALL
and
CHRISTOPHE PLANAS

European University Institute, Florence

EUROPEAN UNIVERSITY INSTITUTE, FLORENCE

ECONOMICS DEPARTMENT

EUI Working Paper **ECO** No. 94/34

Estimation Error and the Specification of Unobserved Component Models

AGUSTÍN MARAVALL
and
CHRISTOPHE PLANAS

BADIA FIESOLANA, SAN DOMENICO (FI)

Printed in Italy in September 1994
European University Institute
Badia Fiesolana
I – 50016 San Domenico (FI)
Italy

Estimation Error and the Specification of Unobserved Component Models

by

Agustín MARAVALL and Christophe PLANAS

European University Institute, Florence*

June 1994

Abstract

The paper deals with the problem of identifying stochastic unobserved two–component models, as in seasonal adjustment or trend–cycle decompositions. Solutions based on the properties of the component estimation errors are considered, and analytical expressions for the variances and covariances of the different types of estimation errors (errors in the final, preliminary, and concurrent estimator and in the forecast) are obtained for any admissible decomposition. These expressions are relatively simple and straightforwardly derived from the ARIMA model for the observed series.

It is shown that, in all cases, the estimation error variance is minimized at a canonical decomposition (i.e., at a decomposition with one of the components noninvertible), and a procedure to determine that decomposition is presented. On occasion, however, the most precise final estimator is obtained at a canonical decomposition different from the one that yields the most precise concurrent estimator.

Three examples illustrate the results and the computational algorithms. The first and second examples are based on the so–called Structural Time Series Model and ARIMA Model Based approaches, respectively. The third example is a class of models often encountered in actual time series.

Key Words: Seasonal Adjustment; Unobserved Component Models; Signal Extraction; ARIMA Models; Identification; Estimation Error

*Department of Economics, Badia Fiesolana, I–50016 San Domenico di Fiesole (FI), ITALY, Tel.: +39-55-46.85.347, Fax: +39-55-46.85.202

Contents

0 Introduction and Overview

We consider the problem of decomposing an observed series into the sum of two uncorrelated components, each one the output of a linear stochastic process, which can be parametrized as an ARIMA model. Thus the basic model (presented in Section 1) is that of an observed ARIMA model with unobserved ARIMA components. Examples are the trend–plus–cycle decomposition often used in business cycle analysis, the seasonally adjusted series plus seasonal component decomposition of economic series and, in general, signal–plus–noise type of decompositions.

The analysis centers on Minimum Mean Squared Error (MMSE) estimators of the unobserved components. Broadly speaking, two main approaches have been developed. In one of them, the overall ARIMA model for the observed series is specified following the standard Box and Jenkins (1970) procedure, and the models for the components are derived from the overall model. This approach has been termed the "ARIMA–Model–Based" (AMB) approach; it has been mostly developed in the context of seasonal adjustment, and basic references are Burman (1980) and Hillmer and Tiao (1982). The second approach directly specifies the models for the components; it has been termed "Structural Time Series Model" (STSM) approach and basic references are Engle (1978) and Harvey (1989). This approach has been heavily used in applied econometrics work.

But whatever the approach, it is well known that the general unobserved components model presents an important identification problem, which stems from the fact that, for a given series, there is in general an amount of white–noise variation that can be arbitrarily allocated between the two components (see, for example, Bell and Hillmer, 1984; or Watson, 1987). This identification problem is discussed in Section 2. The two assumptions most often used to identify a unique decomposition are: (a) restricting the order of the moving average polynomial in the component models, and (b) assigning all possible noise to one of the components, so as to make the other one noninvertible. In this last case, the decomposition is termed "canonical", and the associated noise–free component, a canonical component. Assumption (a) is typically employed in the STSM approach, while assumption (b) characterizes the AMB approach, where the seasonal component is made noninvertible.

Be that as it may, the fact remains that there is no universally accepted criterion to reach identification in unobserved component models, and the properties of the different admissible decompositions have not been much explored. In this paper, we analyse some of these properties, mostly in connection with the components estimation error. Burridge and Wallis (1985) within the STSM approach, and Hillmer (1985) within the ABM approach, have provided algorithms for computing the variance of the components estimation error. In this paper, an alternative approach, close to the one in Watson (1987), is followed, which permits us to obtain relatively simple analytical expressions for the variances of the components estimation error for different admissible decompositions.

It is argued that, when comparing two admissible decompositions that only differ in the allocation of white noise to the components, the one that yields the most precise estimators should be preferable. There are, however, several types of estimators, depending on the available information. For periods close to the end of the series, preliminary estimators have to be used, which will be revised as new observations become available, until the final or historical estimator is obtained. Since it seems reasonable that an agency producing seasonally adjusted data, for example, would like to provide historical series as precise as possible, we begin by considering (Sections 3 and 4) the historical estimator.

Several properties of the historical estimator and its associated error are derived. In particular, it is shown that the crosscovariance–generating function between the estimators of the two components is identical to the autocovariance–generating function of each component estimation error. Thus the admissible decomposition that minimizes the components estimation error minimizes also the covariance between the two component estimators. Given that the components are assumed orthogonal, this additional feature seems a rather desirable property of the chosen decomposition.

For a given overall ARIMA model, the different admissible decompositions can be expressed as a function of a parameter α in the unit interval. The two extreme values, $\alpha = 0$ and $\alpha = 1$, correspond to the two possible canonical decompositions, each one associated with noninvertibility of one of the components. Section 4 expresses the variance of the final estimation error as a second–order polynomial in α, with the coefficients determined from the overall ARIMA model. The decomposition that yields the most precise component estimators is derived and it is shown that it will always be a canonical one. Which of the two canonical decompositions happens to be depends on the stochastic properties of the series and an easy–to–implement algorithm to determine which component should be made canonical is provided. Heuristically, the rule can be interpreted as making noninvertible the most stable of the two components.

The results are next extended to preliminary estimators and forecasts of the components. Since, for an agency involved in short–term policy, minimizing the error in the measurement of the signal for the most recent period seems an important feature, we first consider (Section 6) the error in the concurrent estimator of the components. This error is the sum of the final estimation error and a "revision" error. The variance of the latter is seen to be, again, a second–order polynomial in the parameter α. Yet, now, the variance of the revision will always be maximized at a canonical decomposition. When the two errors are summed, however, it is shown that it will still be the case that a canonical decomposition always minimizes the concurrent estimation error.

Section 7 extends the results to any preliminary estimator of the components. In all cases, the variance of the estimation error is a polynomial of degree 2 in α, and it is seen how the coefficients are straightforward to derive simply from the overall ARIMA model. The variance is always minimized at a canonical decomposition. It will often be the case that the same canonical decomposition minimizes the variance of the different types of estimators and, broadly, that decomposition will be the one with the most stable component made noninvertible. There are, however, cases, when the components have similar degrees of stability, where the solutions "switch" and, for example, one of the canonical decompositions yields the most precise final estimator, while the other one yields the most precise concurrent estimator. Still, the switching of solutions is seen to happen when the estimation error variances for the two canonical decompositions are relatively close, and hence the choice matters little.

Sections 5 and 8 present three examples. The first one is a "trend–plus–cycle" model similar to the ones used by economists in business–cycle analysis. The second example is a quarterly ARIMA model. The two examples are used to illustrate the identification problem, the derivation of the models for the canonical decompositions, the algorithm that provides the specification with minimum estimation error variance for the different types of estimators, and the computation of the coefficients of the polynomials that express those variances as a function of α. The two examples illustrate the STSM and the AMB approaches. The third example consists of a class of models that are often found to approximate reasonably well the stochastic properties of many series: the so–called Airline Model of Box and Jenkins (1970, chapter 9). This example extends the results in Hillmer (1985), and presents some stylized facts often found in actual time series.

1 The Model

We consider the problem of decomposing an observed series x_t into two Unobserved Components (UC), s_t and n_t, as in

$$x_t = s_t + n_t \tag{1.1}$$

The two components are the output of the linear stochastic processes

$$\phi_s(B)\ s_t = \theta_s(B)\ a_{st}, \tag{1.2.a}$$

$$\phi_n(B)\ n_t = \theta_n(B)\ a_{nt}, \tag{1.2.b}$$

where $\phi_\bullet(B)$ denotes a finite polynomial in the lag operator B, having all roots on or outside the unit circle. Letting $\delta_\bullet(B)$ represent the stationary transformation of the component, we shall also use the representation

$$\phi_s(B) = \varphi_s(B)\ \delta_s(B); \quad \phi_n(B) = \varphi_n(B)\ \delta_n(B), \tag{1.3}$$

where $\varphi_\bullet(B)$ contains the roots outside the unit circle and $\delta_\bullet(B)$ contains the unit roots. Finally, $\theta_\bullet(B)$ denotes a finite polynomial in B with the roots on or outside the unit circle. The model consists of equation (1.1)–(1.2) and some additional assumptions.

Assumption 1: The variables a_{st} and a_{nt} are independent normally distributed white–noise innovations in the components. ∎

Since the component is unobservable, we shall refer to a_{st} and a_{nt} as the "pseudo–innovations". Assumption 1 implies, of course, that the two components are uncorrelated. Important examples of the decomposition (1.1) are the "trend + detrended series" decomposition often used in business cycle analysis, where the trend may be a random walk and the detrended series a low–order stationary process, and the "seasonal component + seasonally adjusted series" decomposition, where the seasonal component is typically modeled as

$$U(B)\ s_t = \theta_s(B)\ a_{st}, \tag{1.4}$$

with $U(B)$ the nonstationary "seasonal" polynomial $U(B) = 1 + B + \ldots + B^{\tau-1}$ (τ denotes the number of observations per year), and the seasonally adjusted series is given by a process of the type:

$$\nabla^d\ n_t = \theta_n(B)\ a_{nt}, \tag{1.5}$$

with $d = 1, 2, 3$. Since, as the examples illustrate, each component is basically characterized by its autoregressive (AR) roots, AR roots associated with different frequencies should be allocated to different components. Thus we specify the following assumption, which also avoids redundant roots in the polynomials of (1.2.a) and (1.2.b).

Assumption 2: The polynomials $\phi_s(B)$ and $\phi_n(B)$ share no root in common. The same holds true for the polynomials $\phi_s(B)$ and $\theta_s(B)$, and for the polynomials $\phi_n(B)$ and $\theta_n(B)$. ∎

Equations (1.1) and (1.2), and Assumptions 1 and 2 imply that the observed series x_t follows the general ARIMA process

$$\phi(B)\, x_t = \theta(B)\, a_t. \tag{1.6}$$

The AR polynomial $\phi(B)$ is given by

$$\phi(B) = \phi_s(B)\, \phi_n(B), \tag{1.7}$$

and hence it can also be factorized as $\varphi(B)\, \delta(B)$, with $\varphi(B) = \varphi_s(B)\, \varphi_n(B)$, and $\delta(B) = \delta_s(B)\, \delta_n(B)$, so that $\delta(B)$ denotes the stationarity–inducing transformation for x_t. The Moving Average (MA) part, $\theta(B)\, a_t$, is determined by the identity:

$$\theta(B)\, a_t = \phi_n(B)\, \theta_s(B)\, a_{st} + \phi_s(B)\, \theta_n(B)\, a_{nt}, \tag{1.8}$$

and the constraint that the roots of $\theta(B)$ lie on or outside the unit circle. We shall require however that $\theta(B)\, a_t$ be invertible; this is guaranteed with the following assumption.

Assumption 3: The polynomials $\theta_s(B)$ and $\theta_n(B)$ share no unit root in common. ∎

Without loss of generality, and unless otherwise specified, throughout the paper it will be assumed that $V_a = 1$, where V_a is the variance of a_t in (1.6). It should be kept in mind, thus, that the variance of the pseudo–innovations, V_s and V_n, will be implicitly expressed as a fraction of V_a.

2 Identification of the Model

Having observations on x_t, model (1.6) can be identified from the data. For the rest of the discussion, we shall assume that the ARIMA model for x_t is known. Given this overall model, there is obviously an infinite number of ways of decomposing x_t as in (1.1)–(1.2) under Assumptions 1–3.

If the only identification restrictions that are considered are restrictions in the orders of the polynomials of (1.2), then the necessary and sufficient condition for model identification is that the order of the AR polynomial for at least one of the components be larger than the order of the MA polynomial; see Hotta (1989). Thus, letting p_s, p_n, q_s, and q_n denote the orders of the polynomials $\phi_s(B)$, $\phi_n(B)$, $\theta_s(B)$, and $\theta_n(B)$, respectively, under

Assumption 4a: $p_s > q_s$ or $p_n > q_n$ (or both), ∎

the model consisting of equations (1.1)–(1.2) and Assumptions 1, 2, and 3, is identified.

Be that as it may, since the model for a trend component is not an objective reality that one attempts to capture, but rather a tool designed by the analyst, one may question whether zero–coefficient restrictions are the only constraints that should be imposed. To illustrate the point, we consider an example consisting of a simple UC model similar to the ones used in business cycle analysis (see, for example, Stock and Watson, 1988, 1991). The observed (annual) series is the sum of a trend component, s_t, and a detrended series, n_t, where the trend is the random–walk process

$$\nabla\, s_t = a_{st}, \tag{2.1.a}$$

and the detrended series is the stationary ARMA(1, 1) model

$$(1 + .7\, B)\, n_t = (1 + .2B)\, a_{nt}. \tag{2.1.b}$$

Direct inspection of (2.1.b) shows that the detrended series consists of a stationary cyclical behavior (with period 2) and some purely random noise. Assumptions 1–3 are assumed to hold, and the equations in (2.1) imply that the observed series x_t can be seen as the output of the ARIMA (1, 1, 2) process:

$$(1 + .7B)\, \nabla\, x_t = \theta(B)\, a_t. \tag{2.2}$$

Setting, for our example, $V_s = 5V_n$, it is easily found that $\theta(B) = (1 + .364B - .025B^2)$. For a time series generated by (2.1), Figures 1a and 1b display the two components, and Figures 2a and 2b exhibit the spectra of x_t and of the two components, which we shall represent as $g_x(\omega)$, $g_s(\omega)$, and $g_n(\omega)$, with ω being the frequency in radians. To simplify terminology, "spectrum" will also denote the pseudospectrum of nonstationary series (see Harvey, 1989). Figure 2b shows that $g_s(\omega)$ has a minimum for $\omega = \pi$, which is found to be equal to $g_s(\pi) = V_s/4$. It follows that if a white–noise component u_t, with variance V_u in the interval $[0, V_s/4]$, is removed from s_t and added to n_t, the resulting components also provide a perfectly acceptable decomposition of x_t. The only difference would be that the new s_t component would be smoother, while n_t would now be noisier. This is clearly evidenced in Figures 1c and 1d, which display the component series when white noise with $V_u = V_s/5$ is transferred from s_t to n_t.

In general, if white noise with variance $0 \leq V_u \leq V_s/4$ is removed from s_t and assigned to n_t, it is straightforward to find that the new s_t and n_t components follow processes of the type:

$$\nabla\, s_t = (1 + \theta_s B)\, a_{st} \tag{2.3.a}$$

$$(1 + .7B)\, n_t = (1 + \theta_n B)\, a_{nt}, \tag{2.3.b}$$

For a given model (2.2) for the observed series, different decompositions of the type (2.3) would provide admissible decompositions that would differ in the way the noise contained in the series is allocated to the two components.

Consider an analyst interested in whatever is in the series that cannot be attributed to the trend. He wishes, thus, to remove the trend and nothing but the trend. He will, consequently, avoid adding noise to the trend component, and would choose the decomposition for which V_u is equal to its maximum value $V_s/4$. By choosing this solution, the component s_t (and hence n_t) becomes identified. (Identification of unobserved components by using the "minimum extraction" principle was first proposed by Box, Hillmer, and Tiao, 1978; and Pierce, 1978.) The spectra of the last two components are given in Figures 2c and 2d, where they are compared to the spectra of the components in Figure 1. Since the requirement that it should not be possible to decompose s_t into a smoother component plus white noise implies that $g_s(\pi) = 0$, and since the time domain equivalent of this spectral zero is the presence of the factor $(1+B)$ in the MA part of the component model, s_t will follow the noninvertible model

$$\nabla\, s_t = (1 + B)\, a_{st},$$

and the model for n_t will be as in (2.3.b).

Alternatively, a similar type of reasoning may lead to the transfer of noise from n_t to s_t. Assume, for example, that model (2.2) holds for a time series observed with a twice–a–year frequency. Then model (2.3.b) represents a seasonal component and, if interest centers on the seasonally adjusted series, one may wish to remove from the series as little as possible, and hence the chosen decomposition would consist of a noninvertible seasonal component n_t, with $g_n(0) = 0$, and an invertible seasonally adjusted series s_t. As a consequence, the seasonal component would follow the model

$$(1 + .7B)\, n_t = (1 - B)\, a_{nt},$$

and the model for s_t would be as in (2.3.a). Therefore, the minimum extraction requirement yields two canonical solutions, both of which can be easily justified; each one is characterized by noninvertibility of one of the two components.

Back to the general case of (1.2), assume, in general, that s_t is an invertible and identified component (i.e., $p_s > q_s$). Then, a white–noise component can be removed from s_t and assigned to n_t. It is easily seen that the new model for s_t has $p_s = q_s$; thus we replace Assumption 4a with the more general one

Assumption 4b: $p_s \geq q_s$ or $p_n \geq q_n$ (or both). ■

For a given ARIMA model for the observed variable, the class of admissible decompositions is given by the pair of components s_t and n_t satisfying (1.1), (1.2), (1.7), (1.8), and Assumptions 1, 2, 3, and 4b. We require, of course, nonnegative spectra $g_s(\omega)$ and $g_n(\omega)$. In the general case of an infinite number of admissible decompositions, identification of a unique model can then be reached with the following assumption:

Assumption 5: For $\omega \in [0, \pi]$, either $\min g_s(\omega) = 0$ or $\min g_n(\omega) = 0$ (or both). ■

Identification is, in this case, obtained by forcing a component to be noninvertible. Following Box, Hillmer, and Tiao (1978), a noninvertible component will be denoted a "canonical" component, and the associated decomposition, a canonical decomposition. Since the spectra of the component cannot be negative, in the two–component case, there will be two canonical decompositions. One of them puts all additive white noise in the component n_t, the other one, in the component s_t. Any admissible decomposition can be seen as something in between, whereby some noise is allocated to n_t and some to s_t.

As shown in Hillmer and Tiao (1982), canonical components display some important features. In particular, any other admissible component is equal to the canonical one plus added noise, and hence the canonical requirement makes the component as smooth as possible. On the negative side, Maravall (1986) shows how canonical components can produce large revisions in the preliminary estimators of the component. Besides, the existence of two canonical solutions reflects some basic ambiguity concerning the desirable properties of a component. It seems reasonable, for example, that, in order to avoid noise–induced overreaction, the monetary authority may be interested in a smooth (noise–free) seasonally adjusted series. On the other hand, it sounds also reasonable that the analyst wishes to keep in the series everything but seasonality, in which case the seasonal component would be noise–free. Therefore, both canonical solutions could, in principle, be rationalized.

Some additional suggestions have been made to overcome uncertainty over which admissible decomposition should be chosen. For example, given that different admissible decompositions imply different properties of the estimators, Watson (1987) and Findley (1985) propose to select the one that has maximum mean–square estimation error. Be that as it may, as a general rule, canonical components (i.e., Assumptions 4b and 5) are used in the AMB approach, while zero–coefficient restrictions (i.e., Assumption 4a) are used in the STSM approach. This latter type of assumption is typically found in econometric applications of UC models. Besides its simplicity, the choice may possibly reflect the tradition in econometrics of identifying models (in particular, simultaneous equation models) by using zero–coefficient restrictions (see, for example, Theil, 1971).

3 MMSE Estimators and Their Properties

3.1 Optimal Estimators of the Components

We have mentioned that the properties of the component estimator will depend on the admissible decomposition selected. Our intention is to explore this dependence. In order to do that, we consider first the case of a complete realization of the process, i.e., the case of a series x_t with t going from $-\infty$ to ∞. Let the series be stationary, and write (1.2) and (1.6) more compactly as

$$x_t = \psi(B)\, a_t; \qquad s_t = \psi_s(B)\, a_{st}; \qquad n_t = \psi_n(B)\, a_{nt}, \tag{3.1}$$

where $\psi(B) = \theta(B)/\phi(B)$, $\psi_s(B) = \theta_s(B)/\phi_s(B)$, and $\psi_n(B) = \theta_n(B)/\phi_n(B)$. The minimum Mean Squared Error (MSE) estimator of s_t is given by

$$\hat{s}_t = \nu(B, F)\, x_t = V_s\, \frac{\psi_s(B)\, \psi_s(F)}{\psi(B)\, \psi(F)}\, x_t, \tag{3.2}$$

where F is the forward operator $F = B^{-1}$; see Whittle (1963). The symmetric and centered filter $\nu(B, F)$ is the so–called Wiener–Kolmogorov (WK) filter. Letting $A_j(B, F)$ denote the AutoCovariance Generating Function (ACGF)

$$A_j(B, F) = \psi_j(B)\, \psi_j(F)\, V_j, \qquad j = x, s, n,$$

where we use the convention $\psi_x(B) = \psi(B)$, $V_x = V_a = 1$, expression (3.2) can be rewritten

$$\hat{s}_t = [A_s(B, F)/A_x(B, F)]\, x_t. \tag{3.3}$$

In terms of the AR and MA polynomials, after simplification, the WK filter can be expressed as:

$$\nu(B, F) = V_s\, \frac{\theta_s(B)\, \theta_s(F)\, \phi_n(B)\, \phi_n(F)}{\theta(B)\, \theta(F)} \tag{3.4}$$

Expression (3.4) shows that, under Assumption 3 (invertible observed series), the filter will be convergent, independently of the roots of the AR polynomials. The filter (3.4) in fact extends to nonstationary series, with unit roots in $\phi_s(B)$ and/or $\phi_n(B)$; see Bell (1984), and Maravall (1988). The WK filter (3.4) is simply the ACGF of the model

$$\theta(B)\, z_t = \theta_s(B)\, \phi_n(B)\, b_t, \tag{3.5}$$

with b_t white noise with variance V_s. Since $\theta(B)$ is invertible, the model is stationary and its ACGF will converge. The effect on the filter of different admissible decompositions will show up in the MA part of (3.5), through the polynomials $\theta_s(B)$ and $\phi_n(B)$ and the variance V_s.

Unless the model for the series is a pure AR model, the filter (3.4) will extend from $-\infty$ to ∞. Its convergence however guarantees that, in practice, it can be approximated by a finite filter, and it is generally the case that, for the usual series length, the estimator of the component for the central periods of the series can be safely seen as generated by the WK filter (3.4). This estimator, obtained with the complete filter, is often denoted "historical" or "final" estimator; it shall be the one of interest until Section 6.

3.2 Covariance Between Estimators

It is a well–known result that minimum MSE estimators of orthogonal components yield estimators with nonzero crosscovariances. This discrepancy has been the cause of concern (see, for example, Nerlove, 1964; Granger, 1978; and Garcia Ferrer and Del Hoyo, 1992), and hence one could argue that a desirable identification criterion would be to select, among the admissible decompositions, the one that minimizes the (lag–0) covariance between the estimators. This covariance is easily found from the following result.

<u>**Lemma 1**</u>: Let $C(B, F)$ denote the CrossCovariance Generating Function (CCGF) for the two estimators $\hat{s}_t$ and $\hat{n}_t$. Then $C(B, F)$ is equal to the ACGF of the model

$$\theta(B)\, z_t = \theta_s(B)\, \theta_n(B)\, b_t, \tag{3.6}$$

where b_t is white noise with variance $(V_s\, V_n)$. ∎

<u>**Proof**</u>: Combining (3.4), (1.6), and (1.7), it is possible to express the estimator $\hat{s}_t$ in terms of the innovations a_t of the model for the observed series. After simplification, it is found that

$$\phi_s(B)\, \hat{s}_t = \theta_s(B)\, \alpha_s(F)\, a_t, \tag{3.7}$$

where $\alpha_s(F)$ is the (convergent) forward filter

$$\alpha_s(F) = V_s\, \frac{\theta_s(F)\, \phi_n(F)}{\theta(F)}. \tag{3.8}$$

An equivalent expression is found for $\hat{n}_t$ by simply interchanging the subindices s and n. Combining the two expressions and cancelling common factors, it is obtained that

$$C(B, F) = (V_s\, V_n)\, \frac{\theta_s(B)\, \theta_n(B)\, \theta_s(F)\, \theta_n(F)}{\theta(B)\, \theta(F)}, \tag{3.9}$$

which is the ACGF of model (3.6). ∎

Lemma 1 implies that $C(B, F)$ is symmetric and convergent. Since model (3.6) is stationary, all covariances will be finite. The variance of the model yields the lag–0 covariance between $\hat{s}_t$ and $\hat{n}_t$; this covariance, thus, will always be positive (and the variance of the estimator always underestimates the variance of the component). However, the fact that the covariances between $\hat{s}_t$ and $\hat{n}_t$ are finite implies the following result.

Lemma 2: When the series x_t is nonstationary, the estimators $\hat{s}_t$ and $\hat{n}_t$ are uncorrelated. ∎

For nonstationary series (the case of applied interest) minimum MSE estimation of the components preserves, thus, the orthogonality assumption, and, for example, the statement in García Ferrer and Del Hoyo (1992) that "whereas the theoretical components are uncorrelated, the estimators will be correlated in general" is only correct for stationary series. Further, it is easily found from (3.7) and (3.8) and the equivalent expressions for n_t, that, although the estimators $\hat{s}_t$ and $\hat{n}_t$ are uncorrelated, certain linear combinations of them — namely, the stationary transformations $\delta_s(B)\ \hat{s}_t$ and $\delta_n(B)\ \hat{n}_t$ — are correlated.

It is worth pointing out an interesting feature of the estimators of nonstationary trend and seasonal components. Although both are nonstationary series which, moreover, cannot be cointegrated (since the unit AR roots are different), they display stationary crosscovariances. Thus, the two estimators diverge in time, each one with a nonstationary variance, but their crosscovariances remain constant.

Back to the covariance between the component estimators, model (3.6) shows that different admissible decomposition would affect its MA part, through $\theta_s(B)$, $\theta_n(B)$, V_s and V_n. But before we look at which admissible decomposition minimizes the covariance between the estimators, let us turn our attention to another possibly desirable feature of the estimators.

3.3 The Error in the Component Estimator

The error in the UC estimator depends on the particular admissible decomposition selected and, clearly, a small estimation error is a desirable property of an estimator. Since the data do not discriminate among admissible decompositions, the selection of a particular one reflects a choice of the analyst. In the absence of a compelling reason to select a particular decomposition, why not choose the one that provides the most precise estimator of the component? Since the error in $\hat{s}_t$ is equal to that in $\hat{n}_t$, minimizing both estimation errors seems an attractive feature of the selected model.

To see the dependence of the estimation error on the admissible decomposition chosen, we use the following Lemma.

Lemma 3: Let e_t denote the estimation error $e_t = s_t - \hat{s}_t = \hat{n}_t - n_t$. Then e_t can be seen as the output of the ARMA model

$$\theta(B)\ e_t = \theta_s(B)\ \theta_n(B)\ d_t, \tag{3.10}$$

where d_t is a white noise with variance $(V_s \, V_n)$. ∎

<u>**Proof**</u>: The Lemma is a straightforward application of Theorem 3 in Pierce (1979), for the case $\delta_c(B) = 1$ and $V_a = 1$. ∎

From Lemmas 1 and 3, the following result is trivially obtained

<u>**Lemma 4**</u>: The ACGF of e_t is equal to the CCGF betweem $\hat{s}_t$ and $\hat{n}_t$. ∎

<u>**Corollary 1**</u>: The admissible decomposition with minimum estimation error of the components minimizes also the covariance between the two component estimators. ∎

We turn our attention to the identification of the admissible decomposition that exhibits those desirable properties.

4 Estimation Errors and Admissible Decompositions; the Canonical Decomposition Revisited

As mentioned in Section 2, each admissible decomposition is characterized by a particular allocation of the noise to the two components. Let s_t and n_t denote an admissible decomposition of x_t; then $g_x(\omega) = g_s(\omega) + g_n(\omega)$. Let, for $\omega \in [0, \pi]$, $V_u^s = \min g_s(\omega)$, and $V_u^n = \min g_n(\omega)$. The total amount of "additive" noise in x_t that can be distributed between the components is equal to $V_u = V_u^s + V_u^n$. We shall express each admissible decomposition in terms of a parameter α that reflects the particular noise allocation. Denote by s_t^0 and n_t^0 the decomposition with s_t canonical and n_t with maximum noise, and let $g_s^0(\omega)$, $g_n^0(\omega)$, $A_s^0(B, F)$, and $A_n^0(B, F)$ be the associated spectra and ACGFs of the components. These functions, as well as the models for the underlying components, can be derived from the ARIMA model for the observed series (as shall be illustrated in the next section). Since any admissible component s_t^α is equal to s_t^0 plus an amount of noise with variance in the interval $[0, V_u]$, any admissible decomposition, s_t^α and n_t^α, can be expressed as

$$g_s^\alpha(\omega) = g_s^0(\omega) + \alpha \, V_u \tag{4.1.a}$$

$$g_n^\alpha(\omega) = g_n^0(\omega) - \alpha \, V_u \tag{4.1.b}$$

with $\alpha \in [0, 1]$. The two canonical decompositions (one with s_t canonical, the other with canonical n_t) can be seen as the two extreme cases $\alpha = 0$ and $\alpha = 1$.

The time domain equivalent of (4.1) is given by the relationships

$$A_s^\alpha(B,F) = A_s^0(B,F) + \alpha\, V_u \tag{4.2.a}$$

$$A_n^\alpha(B,F) = A_n^0(B,F) - \alpha\, V_u, \tag{4.2.b}$$

and, for any α, $A_x(B,F) = A_s^\alpha(B,F) + A_n^\alpha(B,F)$. Our aim is to derive an expression that relates the variance of the component estimation error, $V(e_t^\alpha)$, to the parameter α. That variance, we recall, is also the covariance between the two component estimators.

<u>**Lemma 5**</u>: Let $e_t^\alpha = s_t^\alpha - \hat{s}_t^\alpha = \hat{n}_t^\alpha - n_t^\alpha$, and denote by $\hat{s}_t^0$ the estimator of s_t^0 (the canonical s_t); that is $\hat{s}_t^0 = \nu^0(B,F)\, x_t$, where $\nu^0(B)$ is the corresponding WK filter. Then,

$$V(e_t^\alpha) = V(e_t^0) + (1 - 2\,\nu_0^0)\, V_u\, \alpha - h_0\, V_u^2\, \alpha^2, \tag{4.3}$$

where e_t^0 is the error in $\hat{s}_t^0$, ν_0^0 is the central weight of the filter $\nu^0(B,F)$, and h_o is the central weight of the filter $h(B,F) = [\psi(B)\,\psi(F)]^{-1}$. ■

<u>**Proof**</u>: From Lemma 4, ACGF $(e_t^\alpha) =$ CCGF $(s_t^\alpha,\ n_t^\alpha)$. Since the latter can be expressed as $(A_s^\alpha(B,F)\, A_n^\alpha(B,F))/A_x(B,F)$, considering (4.2),

$$\text{ACGF } (e_t^\alpha) = [A_s^0(B,F) + \alpha\, V_u]\, [A_n^0(B,F) - \alpha\, V_u]\, [A_x(B,F)]^{-1} = A_s^0(B,F)\; A_n^0(B,F)/A_x(B,F) + [1 - 2A_s^0(B,F)/A_x(B,F)]\; V_u\, \alpha - [1/A_x(B,F)] V_u^2\, \alpha^2.$$

Expression (4.3) is obtained by noticing that, from Lemma 3, $A_s^0(B,F)\; A_n^0(B,F) \,/\, A_x(B,F) =$ ACGF (e_t^0), that $A_s^0(B,F) \,/\, A_x(B,F) = \nu^0(B,F)$, and that $A_x(B,F) = \psi(B)\; \psi(F)$. ■

Lemma 5 expresses the variance of the component estimation error as a second–order polynomial in α, with coefficients that can be obtained from the "observed" ARIMA model. Considering that $V(e_t^0)$ is the variance of model (3.10) for the case of a canonical s_t, ν_0^0 is the variance of model (3.5) for the case of a canonical s_t, and h_0 is the variance of the inverse model of (1.6), given by

$$\theta(B)\; z_t = \phi(B)\; a_t, \tag{4.4}$$

(see Cleveland, 1972), the three coefficients of (4.3) can be easily computed. The three can be seen as the variance of ARMA models with the AR polynomial always equal to $\theta(B)$. Notice that ν_0^0 is the coefficient of x_t in the filter that provides the historical estimator of s_t^0.

From Lemma 5 it is straightforward to find which admissible decomposition minimizes the variance of the component estimation error:

Lemma 6: For $\alpha \in [0,1]$, $V(e_t^\alpha)$ is minimized

(a) at $\alpha = 0$ when $2\,\nu_0^0 + V_u\, h_0 \leq 1$,

(b) at $\alpha = 1$ otherwise. ∎

Proof: Since $0 \leq \alpha \leq 1$ denotes the range of admissible decompositions, $V(e_t^\alpha)$ is positive over that range. Given that h_0 is also positive, (4.3) implies that $V(e_t^\alpha)$ is a concave function of α. It follows that within a finite interval, the minimum of $V(e_t^\alpha)$ will always be at one of the two boundaries. Since $V(e_t^1) - V(e_t^0) = V_u\,(1 - 2\nu_0^0)$ - $V_u^2\ h_0$, under condition (a), $V(e_t^1) \geq V(e_t^0)$ and $\alpha = 0$ will provide the minimum; trivially, $\alpha = 1$ provides the minimum otherwise. ∎

When $2\nu_0^0 + V_u\, h_o = 1$, then $V(e_t^0) = V(e_t^1)$, and both canonical solutions provide the same estimation MSE, and provide thus two minima for $V(e_t^\alpha)$, within the admissible range for α.

As a function of α, $V(e_t^\alpha)$ given by (4.3) is a parabola, positive over the interval $[0,1]$, with a finite maximum for, say, α_m. If α_m is contained in the interval $[0,1]$, then either $\alpha = 0$ or $\alpha = 1$ may minimize $V(e_t^\alpha)$; when $\alpha_m > 1$, the minimum will be for $\alpha = 0$, and when $\alpha_m < 0$, it will be for $\alpha = 1$. Since $\alpha_m = (1 - 2\nu_0^0)/2h_o\, V_u$, it can be easily checked that the three cases are possible.

Lemma 6 implies that the component estimators with minimum MSE and minimum crosscovariance are always found at one of the two canonical decompositions. In order to determine which one of the two provides that minimum, let the estimators corresponding to the two canonical decompositions be given by

a) when $\alpha = 0$ (canonical component is s_t):

$$\hat{s}_t^0 = \nu_s^0(B,F)\, x_t, \tag{4.5.a}$$

$$\hat{n}_t^0 = \nu_n^0(B,F)\, x_t; \tag{4.5.b}$$

b) when $\alpha = 1$ (canonical component is n_t):

$$\hat{s}_t^1 = \nu_s^1(B,F)\, x_t, \tag{4.6.a}$$

$$\hat{n}_t^1 = \nu_n^1(B,F)\, x_t, \tag{4.6.b}$$

where, for example, $\nu_s^0 = \sum_{j=-\infty}^{\infty} \nu_{s,j}^0\ (B^j + F^j)$, and similarly for the other WK filters.

Lemma 7: Either $\nu^0_{s,0} + \nu^1_{s,0} > 1$ or $\nu^0_{n,0} + \nu^1_{n,0} > 1$, or both sums equal 1. ∎

Proof: Since $x_t = \hat{s}^\alpha_t + \hat{n}^\alpha_t = [\nu^\alpha_s(B,F) + \nu^\alpha_n(B,F)]\, x_t$, it follows that

$$\nu^0_{s,0} + \nu^0_{n,0} = 1; \qquad \nu^1_{s,0} + \nu^1_{n,0} = 1.(4.7) \tag{4.7}$$

Assume $\nu^0_{s,0} + \nu^1_{s,0} < 1$. If $\nu^0_{n,0} + \nu^1_{n,0} \leq 1$, then, adding the two inequalities, and considering (4.7), yields $2 < 2$. Thus $\nu^0_{n,0} + \nu^1_{n,0} > 1$. Assume now $\nu^0_{s,0} + \nu^1_{s,0} > 1$, then a symmetric argument shows that it has to be that $\nu^0_{n,0} + \nu^1_{n,0} < 1$. Finally, from (4.7) it is immediately seen that, if $\nu^0_{s,0} + \nu^1_{s,0} = 1$, then $\nu^0_{n,0} + \nu^1_{n,0} = 1$. ∎

Up to now, the two components s_t and n_t have been treated symmetrically, so that, in a particular application, s_t, for example, could denote any of the two components considered. We break now this symmetry and denote by s_t the component with the largest central weight in the associated WK filter that provides the canonical component estimator. Thus, without loss of generality, we assume the following:

Assumption 6a: $\nu^0_{s,0} \geq \nu^1_{n,0}$. ∎

Now it becomes possible to identify which of the two canonical decompositions has minimum estimation error.

Lemma 8: Among all admissible decompositions, under Assumption 6a, the estimator MSE is minimized for the decomposition with canonical n_t. ∎

Proof: The series x_t can always be decomposed as in

$$x_t = s^0_t + n^1_t + u_t, \tag{4.8}$$

where s^0_t and n^1_t are the two canonical components, and u_t is white noise with variance V_u. The WK filters that provide the estimators of s^0_t and n^1_t are (4.5.a) and (4.6.b); the WK filter for u_t is given by

$$\nu_u(B,F) = V_u \, \frac{\phi(B)\,\phi(F)}{\theta(B)\,\theta(F)} \, ,$$

which is equal to the ACGF of the inverse model (4.4), scaled by V_u. It follows that $V_u\, h_0$ is the central coefficient of the WK filter for u_t. Therefore,

$$\nu^1_{s,0} = \nu^0_{s,0} + V_u h_0 \tag{4.9}$$

is the central weight of the WK filter associated with the decomposition that assigns all white noise to s_t (i.e., with the decomposition with n_t canonical).

From Assumption 6a, $\nu_{s,o}^0 \geq \nu_{n,0}^1$ or, adding $\nu_{s,0}^0 + V_u h_0$ to both sides of the inequality,

$$2\nu_{s,0}^0 + V_u h_0 \geq \nu_{n,0}^1 + \nu_{s,0}^1, \tag{4.10}$$

where use has been made of (4.9). From the second equality in (4.7), $2\nu_{s,0}^0 + V_u h_0 \geq 1$, and hence we are in case (b) of Lemma 6. (When Assumption (6a), and hence (4.10), holds as an equality, then the two canonical decompositions provide two identical minima of $V(e_t^\alpha)$.) ∎

Lemma 8 provides a simple procedure to determine which canonical decomposition provides minimum component estimation error (and minimum covariance between the two component estimators). For each of the two components compute the central weight of the WK filter that yields the estimator of the component in its canonical form. Then, set as canonical component the one with the smallest weight. Notice that, from the two canonical specifications, the central weights of the WK filters can be simply computed as the variance of the ARMA model (3.5). Three remarks seem worth adding:

(a) Since ν_0^0 measures the contribution of observation x_t to the component estimator, the precision of the estimator is maximized by assigning all additive noise to the component for which that contribution is largest.

(b) In the important application to seasonal adjustment, if s_t denotes the seasonal component and n_t the adjusted series, it is often the case that $\nu_{s,0}^0 < \nu_{n,0}^0$ and hence the most precise estimates of s_t and n_t are obtained with a canonical seasonal component. In these cases, the "minimum extraction" principle used in the AMB approach to seasonal adjustment provides also the most precise estimators, with minimum crosscovariance.

(c) While one of the two canonical decompositions always provides the most precise estimators, the other may or may not yield estimators with maximum MSE. When $\alpha_m < 0$ or $\alpha_m > 1$, then it maximizes $V(e_t^\alpha)$, and coincides thus with the minimax solution of Watson (1987). For this solution, of course, the covariance between the estimators is also maximized.

It is worth noticing that the two opposite criteria (choosing the admissible decomposition with maximum or with minimum estimation error variance) stem from a "philosophical" difference. While Watson believes that there is a "true" underlying (unknown) seasonal component model among the set of admissible ones, we believe that reality does not provide for a particular allocation of noise among the two components. This allocation is, in essence, arbitrary. In so far as unobserved components, such as trend or seasonality, are to some degree tools designed by the analyst to address certain problems, it makes sense to choose the most precise tool among the admissible ones.

(d) Expression (4.3) corresponds to expression (3.9) in Watson (1987). The difference is due to the fact that Watson considers a fixed filter, while the filter, in our case, is the optimal one for every value of α. The fact that the filter depends on α invalidates the derivation in Watson, and expression (4.3) is obtained instead.

5 Examples (I)

a) First Example: Trend–plus–Cycle Model

We begin with the same example used to illustrate identification in Section 2. The model is that of equation (2.2) with $\theta(B) = (1 + .364B - .025B^2)$, and accepts a "trend–plus–cycle" decomposition, where the admissible decompositions are given by components of the type (2.3). The identity (1.8) becomes:

$$(1 + .364B - .025B^2)\, a_t = (1 + .7B)(1 + \theta_s B)\, a_{st} + (1 - B)(1 + \theta_n B)\, a_{nt}. \quad (5.1)$$

Since (5.1) is an identity among three MA(2) processes, the associated system of covariance equations consists of 3 equations (one for the variance, and one for each of the lag–1 and lag–2 covariances). The unknowns are the 4 parameters θ_s, θ_n, V_s, and V_n, and hence (2.3) is not identified.

As seen before, an easy way to identify the component models is by adding the zero–coefficient restriction $\theta_s = 0$, which yields of course the decomposition (2.1), with $V_s = 5V_u = .621$ (model (2.2) is standardized by setting $V_a = 1$). From this initial decomposition, it is found that $g_s(\omega) = V_s/2(1 - \cos\omega)$, so that for $\omega \in [0, \pi]$, $\min g_s(\omega) = g_s(\pi) = V_s/4 = .155$. Similarly, $g_n(\omega) = V_n(1.04 + .4\cos\omega)/(1.49 + 1.4\cos\omega)$, and hence $\min g_n(\omega) = g_n(0) = .062$. Since the amount of additive noise that can be exchanged between the components is the sum of these two minima, $V_u = .217$.

Starting from the decomposition (2.1), if we substract from $g_s(\omega)$ its minimum .155, the resulting spectra can be easily factorized to obtain the model for the canonical signal. This model is found to be

$$\nabla s_t^0 = (1 + B)\, a_{st}^0, \qquad V_s^0 = .155. \quad (5.2.a)$$

Since the noise removed from s_t is added to n_t, factorizing the spectrum $(g_n(\omega) + .155)$ yields the model for the component n_t^0, associated with the canonical s_t^0; namely

$$(1 + .7B)\, n_t^0 = (1 + .443B)\, a_{nt}^0, \qquad V_n^0 = .301. \tag{5.2.b}$$

From models (2.2) and (5.2), expressions (3.10), (3.5), and (4.4) can be used to compute the variance of the estimation error, $V(e_t^0)$, the central weight of the WK filter for $\hat{s}_t^0$, $\nu_{s,0}^0$, and the coefficient h_0 of Lemma 5. In particular, $V(e_t^0)$, $\nu_{s,0}^0$, and h_0 are the variances of the processes

$$\begin{aligned} \theta(B)\, z_t &= (1 + B)\,(1 - .443B)\, b_t, & V_b &= V_s^0\, V_n^0 = .047, \\ \theta(B)\, z_t &= (1 + B)\,(1 + .7B)\, b_t, & V_b &= V_s^0 = .155, \\ \theta(B)\, z_t &= (1 + .7B)\,(1 - B)\, b_t, & V_b &= V_a = 1, \end{aligned}$$

respectively, where $\theta(B) = (1 + .364B - .025B^2)$ in all cases. This yields $V(e_t^0) = .101$, $\nu_{s,0}^0 = .441$, $h_0 = 1.653$, and, using (4.3), for any admissible decomposition

$$V(e_t^\alpha) = .101 + .026\alpha - .078\,\alpha^2.$$

For $\alpha \in [0, 1]$, $V(e_t^\alpha)$ is plotted in Figure 3. The estimation error variance is seen to be minimized for $\alpha = 1$, that is, for the decomposition with canonical n_t, in which case $V(e_t^1) = .049$. The maximum value of $V(e_t^\alpha)$ is reached for $\alpha_m = .164$, an interior point of the interval $[0, 1]$; therefore, the decomposition (5.2) is not, in this case, a minimax solution.

That the decomposition with minimum estimation error is the one with canonical n_t can also be found more directly through Lemma 8: The decomposition with canonical n_t is found by removing $\min g_n(\omega) = .062$ from $g_n(\omega)$, and adding it to $g_s(\omega)$ in the initial decomposition (2.1). Factorizing the resulting spectra yields the models

$$\begin{aligned} \nabla\, s_t^1 &= (1 - .084)\, a_{st}^1, & V_s^1 &= .739, \\ (1 + .7B)\, n_t^1 &= (1 - B)\, a_{nt}^1, & V_n^1 &= .018. \end{aligned}$$

Proceeding as before, $\nu_{n,0}^1$ is the variance of the model

$$\theta(B)\, z_t = (1 - B)^2\, b_t, \qquad V_b = V_n^1 = .018,$$

equal to .200. Thus, since $\nu_{s,0}^0 = .441 > \nu_{n,0}^1 = .200$, Assumption 6a holds and Lemma 8 can be directly applied. For this example, the MSE of the historical estimators of the two components is minimized when the cycle is made canonical. Notice that, if x_t is a series observed every 6 months, the component n_t represents a seasonal component. The most precise estimator of the seasonally adjusted series would be obtained by removing a canonical seasonal component.

b) Second Example: Quarterly ARIMA Model

We consider the model

$$(1 - B^4)\, x_t = (1 - .5B)\, a_t, \tag{5.3}$$

which is the same example used for illustration by Kohn and Ansley (1986). The AR part of (5.3) can be rewritten $(1 - B^4) = \nabla U$, where $U = 1 + B + B^2 + B^3$, and hence the model can be decomposed into a seasonal component, s_t, and a seasonally adjusted series, n_t, having models of the type

$$\begin{aligned} U\, s_t &= \theta_s(B)\, a_{st} \\ \nabla\, n_t &= \theta_n(B)\, a_{nt}, \end{aligned}$$

where, under Assumption 4b, $\theta_s(B)$ and $\theta_n(B)$ are, in general, of order 3 and 1, respectively.

The identity (1.8) is now given by

$$(1 - .5B)\, a_t = (1 - B)\, \theta_s(B)\, a_{st} + U\, \theta_n(B)\, a_{nt}, \tag{5.4}$$

and there will be, in general, 5 covariance equations associated with this identity. Since there are 6 unknowns ($\theta_{s1}, \theta_{s2}, \theta_{s3}, \theta_n, V_s$ and V_n), the model is not identified. Proceeding as before, we start with an initial decomposition identified with the use of zero–coefficient restrictions. Restricting to 2 the order of $\theta_s(B)$ and to 0 that of $\theta_n(B)$, the system of covariance equations has now 4 equations and 4 unknowns $(\theta_s, \theta_{s2}, V_s, V_n)$. The system, however, is highly nonlinear and a more efficient way to proceed is the following.

Setting $\theta_s(B) = (1 + \theta_{s1} B + \theta_{s2} B^2)$ and $\theta_n(B) = 1$, the Fourier transform of the identity between the ACGF of the left– and right–hand–side of (5.4) yields

$$\begin{aligned} 1.25 - \cos\omega \;\; &= \;\; (g_0 + g_1\, \cos\omega + g_2\, \cos 2\omega)\,(2 - 2\cos\omega) + (4 + 6\cos\omega + \\ & \quad + 4\cos 2\omega + 2\cos 3\omega)\, V_n, \end{aligned} \tag{5.5}$$

where $g_o = (1 + \theta_{s1}^2 + \theta_{s2}^2)\, V_s$, $g_1 = \theta_{s1}(1 + \theta_{s2})\, V_s$, and $g_2 = \theta_{s2}\, V_s$. Using the identity $2\cos(j\omega)\cos\omega = \cos(j-1)\,\omega + \cos(j+1)\,\omega$, operating in (5.5), and equating coefficients in $\cos(j\omega)$, $j = 0, 1, 2, 3$, the following linear system of equations is obtained

$$\begin{aligned} 1.25 &= 2g_0 - g_1 + 4V_n \\ -1 &= -2g_0 + 2g_1 - g_2 + 6V_n \\ 0 &= -g_1 + 2g_2 + 4V_n \\ 0 &= -g_2 + 2V_n, \end{aligned}$$

with solution $g_0 = .656$, $g_1 = .125$, $g_2 = .031$, and $V_n = .016$. Therefore, the initial decomposition is given by

$$g_s(\omega) = \frac{.656 + .125 \cos \omega + .031 \cos 2\omega}{4 + 6 \cos \omega + 4 \cos 2\omega + 2 \cos 3\omega}, \tag{5.6.a}$$

$$g_n(\omega) = .016/(2 - 2 \cos \omega). \tag{5.6.b}$$

From these spectra it is found that, for $\omega \in [0, 1]$, min $g_s(\omega) = g_s(0) = .051$, and min $g_n(\omega) = g_n(\pi) = .004$. Therefore, $V_u = g_s(0) + g_n(\pi) = .055$.

To obtain the canonical decomposition for $\alpha = 0$ (i.e., the decomposition with canonical seasonal), one simply needs to substract $g_s(0)$ from (5.6.a); the denominator of (5.6.a) remains unchanged, and the numerator becomes $(.453 - .180 \cos \omega - .172 \cos 2\omega - .102 \cos 3\omega)$. Factorizing its spectrum, the model for the canonical s_t component is found to be given by

$$U\, s_t^0 = (1 - .501B - .342B^2 - .156B^3)\, a_{st}^0, \qquad V_s^0 = .325. \tag{5.7.a}$$

Adding, in turn, $g_s(0)$ to (5.6.b) and factorizing the resulting spectrum yields the model for n_t^0:

$$\nabla\, n_t^0 = (1 - .578B)\, a_{nt}^0, \qquad V_n^0 = .088. \tag{5.7.b}$$

(An easy to implement and computationally efficient algorithm to factorize a spectrum can be found in Maravall and Mathis, 1994.)

We can now compute $V(e_t^0)$, $\nu_{s,0}^0$, and h_0 of Lemma 5 as the variances of the models

$$\begin{aligned}
(1 - .5B)\, z_t &= (1 - 501B - .342B^2 - .156B^3)\,(1 - .578B)\, b_t, V_b = V_s^0\, V_n^0 = .029,\\
(1 - .5B)\, z_t &= (1 - 501B - .342B^2 - .156B^3)\,(1 - B)\, b_t, \qquad V_b = V_s^0 = .325,\\
(1 - .5B)\, z_t &= (1 - B^4)\, b_t, \qquad V_b = V_a = 1,
\end{aligned}$$

which yields $V(e_t^0) = .042$, $\nu_{s0}^0 = .701$, and $h_0 = 2.5$. Since $2\nu_{s0}^0 + V_u\, h_0 = 1.54 > 1$, according to Lemma 6 the decomposition with minimum estimation error variance is that with a canonical n_t component ($\alpha = 1$). This is easily confirmed in Figure 3, which plots $V(e_t^\alpha)$, from (4.3) equal to

$$V(e_t^\alpha) = .042 - .022\alpha - .008\alpha^2, \tag{5.8}$$

in the interval $\alpha \in [0, 1]$. The minimum is reached for $V(e_t^1) = .013$. Notice that, in this case, the maximum of $V(e_t^\alpha)$ is reached at $\alpha_m < 0$, and hence (s_t^0, n_t^0) represents the admissible decomposition with largest error variance in the component estimator; i.e., the minimax solution.

The model for the canonical n_t component is found by removing from (5.6.b) the constant $\min g_n(\omega) = g_n(\pi) = .004$ and factorizing the resulting spectrum; the model is found to be

$$\nabla n_t^1 = (1+B)\, a_{nt}^1, \qquad V_n^1 = .004.$$

According to (3.5), ν_{n0}^1 is equal to the variance of the model

$$(1-.5B)\, z_t = (1+B)\, U\, b_t, \qquad V_b = .004,$$

so that $\nu_{n0}^1 = .162$. Since $\nu_{s0}^0 > \nu_{n0}^1$, Assumption 6a is satisfied and Lemma 8 confirms that the decomposition with n_t canonical provides the most precise component estimators. Notice that, while in the first example, the most precise components are obtained with a canonical seasonal (or cyclical) component, in the second example they are obtained with a canonical trend. (It can be seen that the result holds when $\theta = .5$ in (5.3) is replaced by any invertible value of θ.)

c) Third Example: The "Airline Model"

We consider a class of models appropriate for monthly or quarterly series that display trend and seasonality. The model is given by the multiplicative ARIMA expression

$$\nabla \nabla_\tau x_t = (1+\theta_1 B)\,(1+\theta_\tau B^\tau)\, a_t, \tag{5.9}$$

where τ is the number of observations per year and, as before, $V_a = 1$ (all variances in the discussion below are expressed, thus, in units of V_a). Following the work of Box and Jenkins (1970), model (5.9) is often referred to as the "Airline Model". On the one hand, it is a model often encountered in practice; on the other hand, it provides an excellent reference example, since the parameters θ_1 and θ_τ are directly related to the stability of the trend and of the seasonal component. In particular, a value of the parameter θ_1 (θ_τ) close to -1 indicates the presence of a stable trend (seasonal) component. For $-1 < \theta_1 < 1$ and $-1 < \theta_\tau < \bar{\theta}$, where $\bar{\theta}$ is a small positive value (see Figure 4), the model accepts a decomposition of the type (4.8); see Hillmer and Tiao (1982). If the two components decomposition is considered, as in (1.1), with s_t denoting the seasonal component and n_t the seasonally adjusted series, then, for an admissible decomposition, the components follow models of the type

$$U s_t^\alpha = \theta_s^\alpha(B)\, a_{st}^\alpha; \qquad \nabla^2 n_t^\alpha = \theta_n^\alpha(B)\, a_{nt}^\alpha,$$

where $\theta_s^\alpha(B)$ and $\theta_n^\alpha(B)$ are, in general, polynomials in B of order $\tau - 1$ and 2, respectively. The two canonical decompositions are associated with $\alpha = 0$ and $\alpha = 1$, and are given by $x_t = s_t^0 + n_t^0 = s_t^1 + n_t^1$, where s_t^0 is the canonical seasonal component and n_t^1 is the canonical seasonally adjusted series or trend.

We have seen earlier that the component estimators with minimum MSE are always obtained with one of the two canonical decompositions. Table 1 presents the estimation error variance associated with the two canonical decompositions for $\tau = 12$ and different values of θ_1 and θ_{12} (within the admissible region). The variance of the error is large for models whose spectra are dominated by the stochastic trend (values of θ_1 close to 1). Inversely, the estimation error variance is small when the model contains relatively stable components.

Table 1: AIRLINE MODEL: Variance of Error in Final Estimator

Theta(1)	Model Spec.	Theta(12) = 0	Theta(12) = -.25	Theta(12) = -.5	Theta(12) = -.75
.75	Canonical s_t	.410	.504	.436	.259
	Canonical n_t	.407	.504	.439	.267
.50	Canonical s_t	.308	.377	.327	.195
	Canonical n_t	.300	.376	.337	.220
.25	Canonical s_t	.226	.274	.239	.144
	Canonical n_t	.210	.271	.255	.190
0	Canonical s_t	.164	.197	.173	.106
	Canonical n_t	.138	.186	.191	.168
-.25	Canonical s_t	.121	.143	.129	.081
	Canonical n_t	.082	.119	.139	.146
-.50	Canonical s_t	.096	.113	.106	.070
	Canonical n_t	.042	.070	.095	.118
-.75	Canonical s_t	.077	.118	.116	.076
	Canonical n_t	.019	.036	.054	.074

s_t: seasonal component
n_t: nonseasonal component

It is further seen that, when the error variance is large, the difference between the two canonical decompositions is relatively small; in that case, which canonical decomposition (and more generally, which admissible decomposition) is chosen has little effect on the precision of the estimator. On the contrary, when the error variance is small, the difference between the two decompositions becomes more pronounced. As an illustration, Figure 3 also contains the estimation error variance as a function of α for the particular case $\theta_1 = -.34$, $\theta_{12} = -.42$, which represent, from the results in Cleveland and Tiao (1976), the Airline Model "closest" to the X11 filters. For $\alpha = 0$ (canonical seasonal), the estimation error variance equals .125 and, for $\alpha = 1$ (canonical trend), it reaches a minimum of .116. These values are somewhat larger than those reported in Burridge and Wallis (1985), using a more complex and accurate approximation to X11.

As seen in Table 1, for some values of θ_1 and θ_{12}, a canonical seasonal yields the most precise estimators, while for other values of θ_1 and θ_{12}, these are obtained using a canonical seasonally adjusted series. Figure 4 presents, in the admissible parameter space, and for the monthly and quarterly model, the line that separates the region where $\alpha = 0$ minimizes the estimation error variance, from that where the minimum is obtained for $\alpha = 1$. It is clearly seen that stable trends imply the use of a canonical seasonally adjusted series (i.e., of a canonical trend), while stable seasonals imply the use of a canonical seasonal. This was to be expected from Assumption 6a and Lemma 8, since more stable components will have smaller central weights in the corresponding WK filter.

6 Admissible Decompositions, Concurrent Estimation and Revisions

Up to now we have considered estimation of the components for an infinite realization of the series. Since the WK filter converges in both directions, as mentioned in Section 3, it can be safely truncated and, for the usual series length, the estimator for the central periods can be seen as the one obtained with the complete filter (the historical or final estimator). It seems reasonable that, for example, a data–producing agency would like to produce historical series as precise as possible, and hence minimizing the error in the final estimator can often be the relevant criterion.

On the other hand, it would seem also reasonable, for someone involved in short–term monitoring or policy–making, to attempt to minimize the error in the component estimator for the most recent period, in order to avoid error–induced policy actions (this concern is certainly present in, for example, the case of monetary policy). Given that for the most recent observation the WK filter cannot be applied, a preliminary estimator has to be used instead. We proceed to consider the error in this preliminary estimator.

Assume that only a finite realization of the series is available. Denote this finite realization by $X_T = [x_1, x_2, \ldots, x_T]$, and by $x_{t/T}$ the forecast of x_t when observations are available up to and including period T. Then, as shown by Cleveland and Tiao (1976), the optimal "preliminary" estimator of s_t is given by

$$\hat{s}_{t/T} = E_T\, s_t = \nu(B,F)\; x^e_{t|T}, \tag{6.1}$$

where $\nu(B,F)$ is the WK filter given by (3.4), and $x^e_{t|T}$ is the series extended with forecasts $x_{T+j/T}$ and backcasts $x_{1-j/T}$, $j = 1, 2, \ldots$. As new observations become available, the forecasts are updated or replaced by the new data and, as a consequence, the estimator of s_t will be revised until it becomes the historical estimator, once the filter has converged.

Among the preliminary estimators of the signal, the one of applied interest is the concurrent estimator of s_t, $s_{T/T}$, which can be written as

$$\hat{s}_{T/T} = \nu_*(B)\, x_{T-1} + \nu_0\, x_T + \nu_*(F)\, x^e_{T+1}, \tag{6.2}$$

where $\nu_*(z)$ is a convergent one–sided polynomial, and x^e_{T+1} denotes the sequence of forecasts $[x_{T+1/T}, x_{T+2/T}, \ldots]$. We shall assume that the series is long enough for the weights of $\nu_*(B)$ to have converged in the direction of the past. In the vast majority of practical applications this is not a restrictive assumption, and it allows us to associate the finite–sample effect on the concurrent estimator with the unavailability of future observations. We can then write the error in the concurrent estimator, $d_T = s_T - \hat{s}_{T/T}$ as

$$d_T = e_T + r_T,$$

where $e_T = s_T - \hat{s}_T$ is the error in the final estimator $\hat{s}_T$ (analysed in Section 3.3), and $r_T = \hat{s}_T - \hat{s}_{T/T}$ is the "revision error" in the concurrent estimator. Under Assumptions 1–3, the two errors, e_T and r_T, are independent (see Pierce, 1980), and this will be true for any admissible decomposition. Different admissible decompositions (i.e., different values of the parameter α), however, will produce different series e_T and r_T. In Section 4 we looked at the dependence of e_T on the parameter α; we now turn our attention to the dependence of the revision error r_T on α.

Let $r^\alpha_t = \hat{s}^\alpha_t - \hat{s}^\alpha_{t|t}$ denote the revision error in the concurrent estimator of s_t when the chosen decomposition is that given by α in (4.1), and let ν^0_0, h_0, and e^α_t be as in Lemma 5. Rewrite expression (3.7) as

$$\hat{s}_t = \xi(B,F)\, a_t = \ldots + \xi_{-1}\, a_{t-1} + \xi_0\, a_t + \xi_1\, a_{t+1} + \ldots, \tag{6.3}$$

where

$$\xi(B,F) = V_s \, \frac{\theta_s(B)\,\theta_s(F)\,\phi_n(F)}{\phi_s(B)\,\theta(F)}. \tag{6.4}$$

The concurrent estimator $\hat{s}_{t|t}$ can be obtained by taking conditional expectations at time t in (6.3). Thus

$$\hat{s}_{t|t} = \ldots + \xi_{-1}\, a_{t-1} + \xi_0\, a_t, \tag{6.5}$$

since $E_t\, a_{t+j} = 0$ for $j \geq 1$. Substracting (6.5) from (6.3), the revision in the concurrent estimator can be expressed as

$$r_t = \sum_{j=1}^{\infty} \xi_j\, a_{t+j}, \tag{6.6}$$

which involves only the coefficients of F^j, $j \geq 1$, in (6.4) and hence is a convergent polynomial that can be truncated after a finite number of terms. For a particular decomposition (and an overall ARIMA model), the ξ–coefficients are easily computed (as shown in Section 8). Expression (6.6), properly truncated, can then be used to compute the ACGF of r_t; in particular

$$V(r_t) \simeq \sum_{j=1}^{M} \xi_j^2, \tag{6.7}$$

where M is the truncation point.

As shown in the Appendix, for the admissible decomposition $x_t = s_t^\alpha + n_t^\alpha$, the dependence of the variance of the revision error in the concurrent estimators of the components on the parameter α can be expressed as follows.

<u>Lemma 9</u>: The variance of the revision error in the concurrent estimator of s_t^α is given by

$$V(r_t^\alpha) = V(r_t^0) + 2(\nu_0^0 - \xi_0^0)\, V_u\, \alpha + (h_0 - 1)\, V_u^2\, \alpha^2, \tag{6.8}$$

where the superscript 0 denotes the value of a parameter when $\alpha = 0$ (i.e., for s_t canonical). ∎

From the ARIMA model for the observed series, we saw how the models for s_t^0 and n_t^0, the components of the canonical decomposition for $\alpha = 0$, can be derived. From these models, expression (6.4) provides the weights $\xi_j^0 (j \geq 0)$, and (6.7) yields $V(r_t^0)$. Thus all the coefficients of $\alpha^j (j = 0, 1, 2)$ in the r.h.s. of (6.8) are determined from the overall ARIMA model. Notice that, writing the concurrent estimator as the one–sided filter

$$\hat{s}_{t|t} = \tilde{\nu}(B)\, x_t,$$

and considering (1.6), it follows that $\xi_0 = \tilde{\nu}_0$. Therefore, ξ_0^0 is the coefficient of x_t in the filter that provides the concurrent estimator of s_t^0.

From Lemma 9, the following Lemma is easily obtained.

<u>**Lemma 10**</u>: For $\alpha \in [0,1]$, $V(r_t^\alpha)$ is maximized:

(a) at $\alpha = 0$ when $2\,\nu_0^0 + (h_0 - 1)\,V_u \leq 2\,\xi_0^0$,

(b) at $\alpha = 1$ otherwise. ■

<u>**Proof**</u>: $V(r_t^\alpha)$ is a polynomial in α of order 2, with positive constant term. Since h_0 is the variance of z_t in (4.4), $h_0 > V_a = 1$, and hence the coefficient of α^2 is always positive. Over the range $\alpha \in [0,1]$, $V(r_t^\alpha)$ is a positive, convex, function and will, as a consequence, display a maximum at one of the boundary values $\alpha = 0$ or $\alpha = 1$. By comparing $V(r_t^0)$ and $V(r_t^1)$, conditions a) and b) of the Lemma are immediately obtained. ■

Lemma 10 has the following implication.

<u>**Corollary 3**</u>: The variance of the revision error in the concurrent estimator (of s_t and n_t) is maximized at one of the two canonical decompositions. ■

Corollary 3 generalizes the result in Maravall (1986), and shows an unpleasant feature of the canonical decompositions: they may imply relatively large revisions in the concurrent estimator of the signal. However, since $V(r_t^\alpha)$ is a convex parabole, it follows that, as was the case for the error in the historical estimator, while one of the two canonical decompositions maximizes the variance of the revision error, it may well be that the other canonical decomposition minimizes that variance. This will happen when α_m, the value of α that minimizes (6.8), falls outside the interval $[0,1]$. The following corollary states the precise condition.

<u>**Corollary 4**</u>: When ξ_0^0 falls outside the interval $[\nu_0^0, \nu_0^0 + (h_0 - 1)\,V_u]$, one of the canonical decompositions minimizes the revision error variance, among all admissible decompositions. ■

Be that as it may, the main concern is not the revision error *per se*, but the total error in the concurrent estimator of the signal. The dependence of the variance of this error on the particular admissible decomposition selected is shown in the following lemma.

<u>**Lemma 11**</u>: The variance of the error in the concurrent estimator of the signal is given by the second–order polynomial in α

$$V(d_t^\alpha) = V(d_t^0) + (1 - 2\xi_0^0)\,V_u\,\alpha - V_u^2\,\alpha^2, \tag{6.9}$$

where d_t^0 is the error that corresponds to the canonical signal. ∎

Proof: Since $V(d_t^\alpha) = V(e_t^\alpha) + V(r_t^\alpha)$, using expressions (4.3) and (6.8), expression (6.9) is obtained. ∎

Expression (6.9) contains the new parameter $V(d_t^0)$, which can also be derived from the overall ARIMA model by noticing that $V(d_t^0) = V(e_t^0) + V(r_t^0)$, where the two terms in the r.h.s. have already been derived.

Lemma 11 allows us to determine which admissible decomposition minimizes the error in the concurrent estimator.

Lemma 12: For $\alpha \in [0, 1]$, $V(d_t^\alpha)$ is minimized

(a) at $\alpha = 0$ when $2\xi_0^0 + V_u \leq 1$

(b) at $\alpha = 1$ otherwise. ∎

When $2\xi_0^0 + V_u = 1$, then the two canonical decompositions display the same minimum, and hence it is irrelevant which one is chosen. The following result is an obvious implication of Lemma 12.

Corollary 5: The variance of the error in the concurrent estimator of the signal is always minimized at one of the two canonical decompositions. ∎

As a consequence, when the effect of the historical estimation error and of the revision error are aggregated, it still remains true that a canonical specification yields the most precise concurrent estimators of the components. Which one of the two canonical decompositions displays that property can be determined through Lemma 12. To get a better insight into the meaning of the conditions stated in the lemma, we proceed as in Section 4, replacing the WK filter $\nu(B, F)$ by the $\xi(B, F)$ filter of expression (6.3), which expresses the estimator as a function of the innovations in the observed series. The relationship between the two filters is given by

$$\xi(B, F) = \nu(B, F)\, \psi(B), \tag{6.10}$$

where $\psi(B)$ has been defined in (3.1). Similarly to expressions (4.5) and (4.6), we can write expression (6.3) for the two canonical decompositions as:

a) Case $\alpha = 0$ (s_t is the canonical component),

$$\begin{aligned} \hat{s}_t^0 &= \xi_s^0\,(B, F)\, a_t, \\ \hat{n}_t^0 &= \xi_n^0\,(B, F)\, a_t, \end{aligned}$$

b) Case $\alpha = 1$ (n_t is the canonical component),

$$\begin{aligned}\hat{s}_t^1 &= \xi_s^1 (B, F)\, a_t, \\ \hat{n}_t^1 &= \xi_n^1 (B, F)\, a_t,\end{aligned}$$

where the coefficient of B^0 in the four filters is ξ_{s0}^0, ξ_{n0}^0, ξ_{s0}^1, ξ_{n0}^1, respectively.

For a white–noise signal u_t, with variance V_u, it is seen that $\xi_u(B, F) = V_u\, \phi(F)/\theta(F)$, so that $\xi_{u0} = V_u$. Considering that, by construction,

$$\hat{s}_t^1 = \hat{s}_t^0 + \hat{u}_t = \left[\xi_s^0(B, F) + \xi_u(B, F)\right]\, a_t,$$

it follows that $\xi_s^1(B, F) = \xi_s^0(B, F) + \xi_u(B, F)$. As a consequence, $\xi_{s0}^1 = \xi_{s0}^0 + V_u$, and Lemma 12 can be stated as

<u>**Lemma 13**</u>: For $\alpha \in [0, 1]$, $V(d_t^\alpha)$ is minimized

(a) at $\alpha = 0$ when $\xi_{s0}^0 + \xi_{s0}^1 \leq 1$,

(b) at $\alpha = 1$ otherwise. ∎

As was the case for the ν–weights in expression (4.7), the ξ–weights satisfy $\xi_{s0}^j + \xi_{n0}^j = 1$, $j = 0, 1$, so that, using the same argument as in the proof of Lemma 7, the following result is obtained.

<u>**Lemma 14**</u>: Either $\xi_{s0}^0 + \xi_{s0}^1 > 1$ or $\xi_{n0}^0 + \xi_{n0}^1 > 1$, or both sums equal 1. ∎

Proceeding as in Section 4, we can replace Assumption 6a with

<u>**Assumption 6b**</u>: In the two component decompositions, let s_t denote the one such that $\xi_{s0}^0 \geq \xi_{n0}^1$. ∎

Then, from Lemmas 13 and 14, the following lemma is obtained similarly to Lemma 8.

<u>**Lemma 15**</u>: Among all admissible decompositions, under Assumption 6b, the one with canonical n_t minimizes the MSE of the concurrent estimators of the two components. ∎

Assumption 6b and Lemma 15 provide an easy alternative way to determine the decomposition that provides the most precise estimator: For each of the two components, compute the weight for B^0 in the filter $\xi^0(B, F)$. Then, select as the decomposition with canonical component the one with the smallest ξ_0^0–weight. Alternatively, since as seen before ξ_0^0 is the coefficient of x_t in the filter that provides the concurrent estimator of s_t^0, the previous results can be stated as follows: to minimize the concurrent estimation error, choose as canonical component the one with smaller first weight in the associated concurrent estimation filter. Since this weight reflects the effect of a_t on the estimator for period t, the criterion is, thus, to assign all additive noise to the component most affected by the concurrent innovation.

As was the case with historical estimation, while one of the two canonical decompositions always minimizes the variance of the error in the concurrent estimator, the other canonical decomposition may or may not maximize that variance. It will maximize the variance when α_m, the value of α that maximizes the function (6.9), falls outside the interval $[0, 1]$. From (6.9) it is easily seen that under Assumption 6b, α will fall outside the unit interval when $\xi_{s0}^0 > .5$; in that case s_t canonical provides the least precise estimator.

Although both the historical and concurrent estimators are most precise when a canonical specification is employed, the canonical specification may well not be the same in both cases. Thus, for example, there are models, as we shall see in the next section, for which the historical seasonally adjusted series is best estimated with a canonical seasonal component, while the concurrent seasonally adjusted series is best estimated with a canonical trend. The switching of solutions is due to the fact that Assumption 6a does not imply Assumption 6b, and viceversa. From Lemmas 6 and 12, the following corollary is immediately obtained.

Corollary 6: Under Assumption 6a, when $\xi_{s0}^0 < (1 - V_u)/2$, the historical estimation error is minimized with a canonical n_t and the concurrent estimation error is minimized with a canonical s_t. Otherwise, n_t canonical minimizes both types of errors.

Under Assumption 6b, replacing ξ_{s0}^0 with ν_{s0}^0, and V_u with $h_0 V_u$ in the above inequality, the same result holds. ∎

7 Extensions and Summary

7.1 Preliminary Estimators

Up to now we have considered the final or historical estimator, and the concurrent one. From an applied point of view these are the estimators of most interest, but the previous results are easily generalized to the case of any other preliminary estimators of the type $\hat{s}_{t|t+k}$ and $\hat{n}_{t|t+k}$ $(k = 1, 2, \ldots)$, intermediate between the concurrent and historical estimator. As before, we assume that the series is long enough for the WK filter to have converged in the direction of the past, so that the error in the preliminary estimator is the sum of the historical estimation error plus the revision error, implied by the fact that the filter has not converged in the direction of the future. In obvious notation,

$$d^{\alpha}_{t|t+k} = s^{\alpha}_t - \hat{s}^{\alpha}_{t|t+k} = e^{\alpha}_t + r^{\alpha}_{t|t+k}, \tag{7.1}$$

where e^{α}_t and $\hat{s}^{\alpha}_t$ are as in Lemma 5 and $r^{\alpha}_{t|t+k} = \hat{s}^{\alpha}_t - \hat{s}^{\alpha}_{t|t+k}$. Proceeding in a similar way as in the case of the revision in the concurrent estimator, and deleting for notational convenience the superscript α, the revision in the preliminary estimator $\hat{s}_{t|t+k}$ can be expressed as

$$r_{t|t+k} = \sum_{j=k+1}^{\infty} \xi_j \, a_{t+j},$$

and its variance can be computed as

$$V(r_{t|t+k}) \simeq \sum_{j=k+1}^{\infty} \xi_j^2 = V(r_t) - \sum_{j=1}^{k} \xi_j^2, \tag{7.2}$$

where $V(r_t)$ is given by (6.7). Denote by $\lambda(B)$ the polynomial

$$\lambda(B) = \psi(B)^{-1} = \sum_{i=0}^{\infty} \lambda_i B^i, \qquad \lambda_0 = 1; \tag{7.3}$$

then, (6.10) can be rewritten $\lambda(B)\xi(B,F) = \nu(B,F)$, from which it is obtained (by equating the coefficients of B^0) that

$$\nu_0 = (\xi_0 + \lambda_1 \xi_1 + \lambda_2 \xi_2 + \ldots) = \sum_{i=0}^{\infty} \xi_i \lambda_i. \tag{7.4}$$

Further, since the inverse ACF of model (1.6) is $\psi(B)^{-1}\psi(F)^{-1}$, it follows that

$$h_0 = \sum_{i=0}^{\infty} \lambda_i^2.$$

Now we can state the lemma that expresses the variance of the error in the preliminary estimator as a function of α (i.e., of the selected admissible decomposition). The proof is in the Appendix.

<u>**Lemma 16**</u>: For $k > 0$, define $\zeta^0_k = \sum_{i=0}^{k} \xi^0_i \lambda_i$ and $\delta_k = \sum_{i=0}^{k} \lambda_i^2$. Then

$$V(d^{\alpha}_{t|t+k}) = V(d^0_{t|t+k}) + (1 - 2\zeta^0_k) V_u \alpha - \delta_k V_u^2 \alpha^2, \tag{7.5}$$

where $V(d^0_{t|t+k})$ is the error in the estimator of the canonical component. ■

The new parameter $V(d^0_{t|t+k})$ can be obtained as $V(e^0_t) + V(r^0_{t|t+k})$, where the second term is obtained plugging in (7.2) the appropriate ξ^0–weights corresponding to the decomposition with $\alpha = 0$. The coefficients of $\alpha^j (j = 0, 1, 2)$ in the r.h.s. of (7.5) are thus fully determined from the overall ARIMA model. Similarly to the cases of Lemmas 5 and 11, the variance of the error is a concave parabole, positive in the interval $\alpha \in [0, 1]$, so that, reasoning in a similar way, Lemma 17 is immediately obtained.

<u>Lemma 17</u>: The error in the preliminary estimator has minimum variance at one of the two canonical decompositions. ■

In order to determine, for a particular value of k, which of the two canonical specifications provides the most precise preliminary estimator, we can proceed as in the cases of Lemmas 5 and 11, and the following result is obtained.

<u>Lemma 18</u>: For $\alpha \in [0, 1]$, $V(d^\alpha_{t|t+k})$ is minimized:

(a) at $\alpha = 0$ when $2\zeta^0_k + \delta_k V_u \geq 1$,

(b) at $\alpha = 1$ otherwise. ■

Notice that, in expression (7.5), when $k \to \infty$, then $\zeta^0_k \to \nu^0_0$, $\delta_k \to h_0$, so that $V(d^\alpha_{t|t+k}) \to V(e^\alpha_t)$; in this case the filter has converged and the estimation error is simply the historical one, with variance given by (4.3). On the other hand, when $k = 0$, then $\zeta^0_k = \xi^0_0$, $\delta_k = 1$, and $V(d^\alpha_{t|t+k})$ becomes the variance of the error in the concurrent estimator, given by (6.9).

It is straightforward to see that condition (a) can also be restated as $\zeta^0_k + \zeta^1_k \geq 1$, where ζ^1_k is obtained with the weights ξ^1_i, corresponding to the specification that assigns all noise to the component s_t. Again, one can proceed to identify which canonical solution provides the minimum estimation error variance by replacing Assumption 6a or 6b with a similar one which uses the coefficients $\zeta^0_{s,0}$ and $\zeta^1_{n,0}$ instead of the ν_0– and ξ_0–coefficients. If s_t denotes the component with the largest ζ_0–coefficient, then $V(d_{t|t+k})$ is always minimized for the decomposition with a canonical n_t component.

The parameter $\zeta^0_k = \sum_{i=0}^k \xi^0_i \lambda_i$ of Lemma 16 turns out to have a very simple and direct interpretation: Considering that the forecast $x_{T|t+k}$, for $T > t+k$, can be expressed as a one–sided filter, say, $x_{T|t+k} = \psi^k(B)\, x_{t+k}$, expression (6.1) can be rewritten as

$$\hat{s}_{t|t+k} = \nu^k(B, F)\, x_t, \tag{7.6}$$

where $\nu^k(B,F)$ is a filter truncated at F^k. Expression (7.6) represents the preliminary estimation filter as applied to the observed series. Let ν_0^k denote the weight of B^0 in this filter, i.e., the weight applied to the observation x_t.

Lemma 19: $\zeta_k^0 = \nu_0^k$. ∎

Proof: Inserting (3.1) in (7.6),

$$\hat{s}_{t|t+k} = \nu^k(B,F)\,\psi(B)\,a_t. \tag{7.7}$$

Taking conditional expectations at time $(t+k)$, expression (6.3) yields

$$\hat{s}_{t|t+k} = \xi^k(B,F)\,a_t, \tag{7.8}$$

where $\xi_k^k(B,F)$ is the filter $\xi(B,F)$ truncated at F^k, and use has been made of the property $E_{t+k}\,a_T = 0$ when $T > t+k$. Comparing (7.7) and (7.8), it is seen that $\nu^k(B,F)\,\psi(B) = \xi^k(B,F)$, or, considering (7.3),

$$\nu^k(B,F) = \xi^k(B,F)\,\lambda(B). \tag{7.9}$$

Equating the coefficients of B^0 at both sides of the identity (7.9),

$$\nu_k^0 = \sum_{i=0}^{k} \xi_i\,\lambda_i. \tag{7.10}$$

For the canonical specification of s_t, (7.10) becomes $\nu_k^0 = \zeta_k^0$. ∎

From the discussion after Lemma 18, and from Lemma 19, the following corollary is obtained.

Corollary 7: Let (1.1) and (1.2) represent the admissible decompositions of a given ARIMA model under Assumptions 1-3. To select the decomposition with smallest MSE in the preliminary estimator of s_t and n_t,

a) compute the weight of x_t (say, ν^0) in the two filters that provide the preliminary estimators of the components specified in their canonical form;

b) choose the canonical decomposition with canonical component the one with the smallest ν^0 weight. ∎

7.2 Forecasts

Since any admissible component can be expressed as the sum of the canonical component plus an orthogonal white–noise component (with variance $\alpha\,V_u$), the forecast of the component will be that of the canonical one plus the forecast of orthogonal white noise. Since the latter will always be zero, it follows that, although different admissible decompositions will provide different historical and preliminary estimators, they will all provide the same forecasts. The standard errors of these forecasts, however, will differ: obviously, they will become larger as αV_u increases. Trivially, thus, the decomposition that minimizes the standard error of the component forecast is that with $\alpha = 0$, that is, when the component itself is canonical. Contrary to the case of estimation erros in current or past signals, the forecasting errors of s_t^α and n_t^α are not the same. The minimum variance forecast error of s_t^α is reached at the canonical decomposition with $\alpha = 0$, while that of n_t^α at the canonical decomposition with $\alpha = 1$. This will hold independently of whether Assumptions 6a or 6b are made, and there is not an admissible decomposition that simultaneously minimizes the forecasting error variance of s_t and n_t. Therefore, knowing which one of the two is the signal of interest, if minimizing the variance of its forecast error is the criterion for choosing an admissible decomposition, then the decomposition selected should have the signal of interest as a canonical component. Still, since all decompositions yield identical forecasts, with the different standard errors reflecting simply the allocation of noise between the components, the previous criterion seems of less interest than minimizing either the historical or the concurrent estimation error.

8 Examples (II)

a) First Example: Trend–plus–Cycle Model

We consider the same example of Section 5a, consisting of an I(1) trend, s_t, and a stationary ARMA(1, 1) cycle, n_t. In order to obtain the variances of the errors in the preliminary estimators (expressions (6.8), (6.9), and (7.5)), we need the parameters $V(e_t^0)$, ν_0^0, V_u, and h_0, which were already computed in Section 5a, plus some of the coefficients of the filter $\lambda(B)$, given by (7.3), and of the filter $\xi^0(B,F)$, given by (6.4). For model (2.2),

$$\lambda(B) = \frac{(1+.7B)\,(1-B)}{1+.364B-.025B^2} = 1 - .664B + \dots,$$

and, for $\hat{s}_t^0$ (the estimator of the canonical seasonal component) the filter $\xi_s^0(B,F)$ becomes

$$\xi_s^0(B,F) = V_s^0\,\frac{(1+B)\,(1+F)\,(1+.7F)}{(1-B)\,(1+.364F-.025F^2)} = V_s^0\,\eta(B,F), \qquad (8.1)$$

where V_s^0 was found to be .155. We wish to express the filter $\eta(B,F)$ as the sum of a filter in B and a filter in F. In order to do that, we first express the numerator and denominator of $\eta(B,F)$ as $(1+B)\,(.7+1.7B+B^2)\,F^2$ and $(1-B)\,(-.025+.364B+B^2)\,F^2$, respectively, and then obtain the partial fractions decomposition:

$$\frac{(1+B)\,(.7+1.7B+B^2)}{(1-B)\,(-.025+.364B+B^2)} = \frac{c_0}{1-B} + \frac{c_1+c_2B+c_3B^2}{-.025+.364B+B^2}. \qquad (8.2)$$

The coefficients c_0, c_1, c_2, and c_3 are easily determined by removing denominators in (8.2), and then equating the coefficients of B^0, B, B^2, and B^3 in the left– and right–hand–side of the resulting identity. This yields the linear system of equations

$$\begin{bmatrix} -.025 & 1 & 0 & 0 \\ .364 & -1 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} .7 \\ 2.4 \\ 2.7 \\ 1 \end{bmatrix}$$

with solution $c_0 = 5.078$, $c_1 = .827$, $c_2 = 1.378$, and $c_3 = -1$. The filter $\eta(B,F)$ can then be expressed as

$$\eta(B,F) = \eta^-(B) + \eta^+(F),$$

where $\eta^-(B) = 5.078\,(1-B)^{-1}$, and $\eta^+(F) = (-1+1.378F+.827F^2)\,(1+.364F-.025F^2)^{-1}$. Multiplying by V_s^0 the coefficients of $\eta^-(B)$ and $\eta^+(F)$, it is found that

$$\begin{array}{rclll} \xi_{sj}^0 & = & .788, & j<0, & \xi_{s0}^0 = .633, \\ \xi_{s1}^0 & = & .270, & \xi_{s2}^0 = .026, & \xi_{s3}^0 = -.003, \\ \xi_{s4}^0 & = & .002, & \xi_{s5}^0 = -.001, & \xi_{sj}^0 \simeq 0, \qquad j>5. \end{array}$$

Setting $M=5$, expression (6.7) yields $V(r_t^0) = .074$, and hence $V(d_t^0) = V(e_t^0) + V(r_t^0) = .175$. We also consider the 1–period revision of the concurrent estimator, so that $k=1$ in Lemma 16. Then, $\delta_1 = 1+\lambda_1^2 = 1.441$, $\zeta_1^0 = \xi_0^0\,\lambda_0 + \xi_1^0\,\lambda_1 = .453$ and, from (7.2), $V(d_{t|t+1}^0) = V(d_t^0) - \xi_1^2 = .103$. Plugging these parameters in expressions (6.8), (6.9), and (7.5), it is obtained that

$$\begin{aligned} V(r_t^\alpha) &= .074 - .083\alpha + .031\alpha^2 \\ V(d_t^\alpha) &= .175 - .057\alpha - .047\alpha^2 \\ V(d_{t|t+1}^\alpha) &= .103 + .020\alpha - .068\alpha^2. \end{aligned}$$

Table 2: TREND–PLUS–CYCLE EXAMPLE: Estimation Error Variance

	Concurrent Estimator	One–period Revision	Final Estimator
canonical seasonal component ($\alpha = 0$)	.175	.103	.101
canonical seasonally adjusted series ($\alpha = 1$)	.070	.055	.049

The three variances are represented in Figure 5, together with the variance of the final estimation error, from Section 5a. For this example, consideration of different estimators does not produce any switching of solutions, and the specification with canonical $n_t(\alpha = 1)$ always minimizes the estimation error variance. (It is straightforward to find that $\xi^1_{n0} = .150$, and hence the conditions of Assumptions 6a and 6b are both met.) The variances of the concurrent, one–period revision, and final estimation errors are given in Table 2.

The use of a canonical n_t component instead of a canonical s_t cuts in less than half the variance of the estimation error, a nonnegligible gain in precision. The variance of the component estimation error for any other admissible decomposition will lie in between the corresponding two previous values.

b) Second Example: Quarterly ARIMA Model

We proceed to analyze the error in the preliminary estimator of the signal for the ARIMA model of Section 5b. As we saw, the model given by (5.3) accepts a decomposition into seasonal component (s_t) and seasonally adjusted series (n_t), where white noise with variance $V_u = .055$ can be exchanged between the two components. In order to compute the variances in expressions (6.8), (6.9), and (7.5), we need the coefficients λ_i of (7.3), the ξ–weights of expression (7.4), and the variances $V(r^0_t)$, $V(d^0_t)$, and $V(d^0_{t|t+k})$, associated with the decomposition with canonical s_t.

As in the previous example, we focus on the one–period revision, and hence need λ_0 and λ_1 in

$$\sum_{i=0}^{\infty} \lambda_i B^i = (1 - B^4)(1 - .5B)^{-1}.$$

Thus $\lambda_0 = 1$, $\lambda_1 = .5$, and δ_k in expression (7.5) becomes $\delta_1 = 1.25$.

To obtain the ξ–weights, we proceed as in the previous example (for a general algorithm, see Maravall, 1994). The polynomials $\theta(B)$, $\phi_n(B)$, $\theta_s(B)$, $\phi_s(B)$, and the variance V_s of expression (6.4), for the case of $\hat{s}_t^0$, are given in (5.3) and (5.7). The polynomial

$$\eta(B,F) = \frac{(1-.501B-.342B^2-.156B^3)}{(1+B+B^2+B^3)} \frac{(1-.501F-.342F^2-.156F^3)(1-F)}{(1-.5F)}$$

can be expressed as

$$\frac{F}{F^4}\,\eta^*(B) = \frac{F}{F^4}\,\frac{(1-501B-.342B^2-.156B^3)}{(1+B+B^2+B^3)}\frac{(-.156-.342B-.501B^2+B^3)(-1+B)}{(-.5+B)}. \tag{8.3}$$

The polynomial $\eta^*(B)$ can then be decomposed as in

$$\eta^*(B) = \frac{c_1+c_2B+c_3B^2}{1+B+B^2+B^3} + \frac{d_1+d_2B+d_3B^2+d_4B^3+d_5B^4}{-.5+B}. \tag{8.4}$$

Removing denominators from (8.3) and (8.4), an identity between two polynomials in B (of order 7) is obtained. Equating the coefficients of B^j, $j = 0, \ldots, 7$, in both sides of the identity yields a linear system of 8 equations in the eight unknowns $c_1, c_2, c_3, d_1, d_2, d_3, d_4$, and d_5. The second term in expression (8.4) can be rewritten as

$$\frac{B^4(d_5+d_4F+d_3F^2+d_2F^3+d_1F^4)}{B(1-.5F)}. \tag{8.5}$$

Combining (8.3) - (8.5), the filter is decomposed as

$$\eta(B,F) = F^3\left(\frac{c_1+c_2B+c_3B^2}{1+B+B^2+B^3}\right) + \left(\frac{d_5+d_4F+d_3F^2+d_2F^3+d_1F^4}{1-.5F}\right). \tag{8.6}$$

The two filters in parenthesis in the right–hand–side of (8.6) are easily obtained (see, for example, the appendix in Box, Hillmer, and Tiao, 1978). The first one is shifted 3 periods, and the sum of the two filters multiplied by V_s^0 yields $\xi(B,F)$. For our purposes, only the weights in F^j, $j \geq 0$, are of interest, namely $\xi_{s0}^0 = .824$, $\xi_{s1}^0 = -.135$, $\xi_{s2}^0 = -.099$, $\xi_{s3}^0 = -.065$, $\xi_{s4}^0 = .018$, $\xi_{s5}^0 = .009$, $\xi_{s6}^0 = .005$, $\xi_{s7}^0 = .002$, $\xi_{s8}^0 = .001$, $\xi_{si}^0 \simeq 0$ $(i > 8)$. Using expression (6.7), $V(r_t^0) = .033$, and hence $V(d_t^0) = V(e_t^0) + V(r_t^0) = .075$. Finally, the parameter ζ_k^0 in expression (7.5) is given by $\zeta_1^0 = \xi_{s0}^0\,\lambda_0 + \xi_{s1}^0\,\lambda_1 = .756$ and, from expression (7.2), $V(d_{t|t+1}^0) = V(d_t^0) - (\xi_{s1}^0)^2 = .060$. Plugging the previous

Table 3: Quarterly Arima Example: Estimation Error Variance

	Concurrent Estimator	One–period Revision	Final Estimator
canonical seasonal component ($\alpha = 0$)	.075	.060	.042
canonical seasonally adjusted series ($\alpha = 1$)	.037	.028	.013

values in expressions (6.8), (6.9), and (7.5), for an admissible decomposition $(s_t^\alpha + n_t^\alpha)$, the variance of the revision in the concurrent estimator, and of the error in the concurrent estimator and its 1–period revision, are given by the functions

$$\begin{aligned} V(r_t^\alpha) &= .033 - .014\alpha + .005\alpha^2 \\ V(d_t^\alpha) &= .075 - .036\alpha - .003\alpha^2 \\ V(d_{t|t+1}^\alpha) &= .060 - .028\alpha - .004\alpha^2. \end{aligned}$$

Figure 6 plots the 3 paraboles in the admissible range $\alpha \in [0, 1]$, together with the variance of the final estimation error given by (5.8). It is seen how the canonical decomposition with n_t canonical ($\alpha = 1$) minimizes all estimation errors, while the decomposition with canonical seasonal component maximizes them. (Again, this result is valid for any invertible value of θ in (5.3).) Table 3 presents the variances of the errors in the concurrent, 1–period revision, and final estimators of the components.

As in the previous example, there is a large gain in the precision of the component estimators when moving from the canonical decomposition with $\alpha = 0$ to the one with $\alpha = 1$. Since the two canonical decompositions represent the maximum and minimum values of the estimation error variance, they represent bounds for the estimation error variance associated with any other admissible decomposition. Finally, compared to the first example, the revision between the concurrent and final estimator now lasts longer: the first–period revision accounts for roughly 40% of the total revision.

c) Third Example: The Airline Model

We proceed to analyze the error in the preliminary estimator of the seasonally adjusted series and of the seasonal component for an admissible decomposition of model (5.9), discussed in Section 5c. Table 4 presents the variance of the error in the concurrent estimator for the two canonical decompositions ($\alpha = 0$ and $\alpha = 1$). Comparing Tables 4 and 1, it is seen that, except for the stable trend–unstable seasonal case, the variance of the final estimation error accounts for (roughly) between 1/3 and 1/2 of the variance of the concurrent estimation error; the revision error is, thus, typically larger than the final estimation error. In the stable trend–unstable seasonal case (θ_1 close to -1, θ_{12} close to 0), the concurrent estimation error becomes small, and the revision accounts for a particularly large fraction. Therefore, the revision becomes relatively more important when the total estimation error is small.

Table 4: AIRLINE MODEL: Variance of Error in Concurrent Estimator

Theta(1)	Model Spec.	Theta(12) = 0	Theta(12) = -.25	Theta(12) = -.5	Theta(12) = -.75
.75	Canonical s_t	1.257	1.151	.905	.521
	Canonical n_t	1.261	1.157	.913	.532
.50	Canonical s_t	.956	.873	.685	.393
	Canonical n_t	.964	.888	.710	.433
.25	Canonical s_t	.699	.641	.505	.292
	Canonical n_t	.710	.665	.551	.369
0	Canonical s_t	.491	.458	.367	.215
	Canonical n_t	.498	.483	.426	.327
-.25	Canonical s_t	.333	.323	.269	.164
	Canonical n_t	.326	.336	.324	.292
-.50	Canonical s_t	.228	.239	.214	.139
	Canonical n_t	.193	.217	.234	.244
-.75	Canonical s_t	.149	.205	.207	.143
	Canonical n_t	.097	.120	.141	.161

s_t: seasonal component
n_t: nonseasonal component

Table 4 shows that when the trend and the seasonal components are unstable (θ_1 close to 1, θ_{12} close to 0), the variance in the concurrent estimation error is large, and, for very unstable components, the concurrent estimator can have an error variance larger than that of the 1–period–ahead forecast of the series. However, as was the case for the final estimation error, when the concurrent estimation error is large, the difference between the two canonical decompositions is small. On the other hand, when the concurrent estimation error is small, the choice of the appropriate canonical decomposition can cut in half the estimation error variance.

Using the same example of Section 5c, with $\theta_1 = -.34$ and $\theta_{12} = -.42$, the parameters of expressions (4.3), (6.8), (6.9), and (7.5) corresponding to the decomposition with a canonical seasonal component, can be derived from the overall ARIMA model in a manner similar to that illustrated in the two previous examples. For an admissible decomposition, the variances of the errors can be expressed as

$$\begin{aligned} V(r_t^\alpha) &= .138 - .018\alpha + .057\alpha^2 \\ V(d_t^\alpha) &= .263 + .081\alpha - .051\alpha^2 \\ V(d_{t|t+12}^\alpha) &= .153 + .065\alpha - .094\alpha^2 \\ V(e_t^\alpha) &= .125 + .099\alpha - .108\alpha^2, \end{aligned}$$

and they are represented in Figure 7. This example illustrates a case of "switching solutions": while the final estimation error is minimized with the decomposition with canonical seasonally adjusted series ($\alpha = 1$), the concurrent estimation error is minimized with the decomposition with a canonical seasonal component. Still, the difference between the errors associated with the two canonical decompositions is relatively small, in particular for the final estimation error case. Table 5 presents the variance of the different estimation errors for the two canonical decompositions; as before, the values are larger than those reported by Burridge and Wallis (1985) for their more accurate approximation to X11.

For different values of θ_1 and θ_τ, Figure 4 also displays the line that separates the region of the admissible parameter space where a canonical seasonal minimizes the concurrent estimation error from that where the minimum is achieved with a canonical seasonally adjusted series (for the monthly and quarterly versions of (5.9)). As with the final estimation error, a stable seasonally adjusted series favors the use of a canonical seasonally adjusted series, and a stable seasonal component favors the use of a canonical seasonal. Compared to the case of the final estimation error, it is seen that, for the concurrent estimation error, the region where a canonical seasonal component provides the most precise estimators has become considerably larger. The area between the two

Table 5: AIRLINE MODEL: Estimation Error Variance

	Concurrent Estimator	12–period Revision	Final Estimator
canonical seasonal component ($\alpha = 0$)	.263	.153	.125
canonical seasonally adjusted series ($\alpha = 1$)	.293	.124	.116

lines represents the models for which the canonical solution that minimizes the concurrent estimation error is different from the one that minimizes the final estimation error, i.e., the region of switching solutions.

9 Summary

In the decomposition of a time series into two orthogonal unobserved components, a basic underidentification problem is the following: Given the ARIMA model for the observed series, there is an infinite number of admissible decompositions that differ in the way the additive noise contained in the series is distributed between the two components. When all the noise is added to one of the components, and hence the other one is noise–free, the decomposition is termed canonical. There are, thus, two canonical decompositions. We have suggested that a desirable property of the particular decomposition chosen among the set of admissible ones is to maximize the precision of the components estimators. We have seen that the minimum estimation error variance is always obtained at one of the canonical decompositions. Which one of the two displays that property depends, however, on the type of estimator considered. This dependence is summarized in Table 6, which indicates which component should be noise–free (canonical) if the variance of the error in the historical estimator, in the concurrent estimator, and in the forecast of the component is to be minimized. The answer is provided under two convenient standardizations of the model (Assumptions 6a and 6b).

Although there are cases in which consideration of different estimators yields the same answer, there are also cases in which different estimators imply

Table 6: Component That Should be Canonical in Order to Minimize the Estimation Error Variance

<table>
<tr><td></td><td>Historical
Estimation Error</td><td>Concurrent
Estimation Error</td><td colspan="2">Forecast Error</td></tr>
<tr><td></td><td>of s_t and n_t</td><td>of s_t and n_t</td><td>of s_t</td><td>of n_t</td></tr>
<tr><td>Under
Assumption 6a
$(\nu^0_{s0} > \nu^1_{n0})$</td><td>$n_t$</td><td>$n_t$ if $\xi^0_{s0} \geq \frac{1-V_u}{2}$
s_t otherwise</td><td rowspan="2">s_t</td><td rowspan="2">n_t</td></tr>
<tr><td>Under
Assumption 6b
$(\xi^0_{s0} > \xi^1_{n0})$</td><td>$n_t$ if $\nu^0_{s0} \geq \frac{1-h_o V_u}{2}$
s_t otherwise</td><td>n_t</td></tr>
</table>

a switch in the canonical specification from one component to the other. The choice would depend, then, on the purposes of the decomposition. One may expect that estimation criteria are likely to be more relevant than forecasting ones. In particular, minimizing uncertainty in the historical series (in which case the covariance between the estimators is also minimized) would appear to us the most important feature.

Be that as it may, leaving aside the trivial case of forecasting, for any type of estimator (concurrent, preliminary, or final), the following result holds: Specify each component in its canonical form and consider the MMSE estimation filter for the component at time t. Let ν_0 denote the coefficient of x_t in this filter. If the component with smallest ν_0 weight is made canonical, then the estimation error variance is minimized (for both components). Thus, if interest centers on having the most precise historical estimator, ν_0 denotes the central weight of the WK filter. If, alternatively, the most precise concurrent estimator is sought, ν_0 denotes the first weight of the one–sided filter. More generally, if interest centers on minimizing the error of the estimator of the component for time t, computed at time $(t+k)$, then ν_0 is the weight of x_t in the truncated filter (i.e., the filter that extends up to x_{t+k}).

Appendix

<u>**Proof of Lemma 7**</u>: From (3.3) and (4.2.a) it is immediately seen that

$$\hat{s}_t^\alpha = [\nu^0(B,F) + h(B,F)\, V_u\, \alpha]\, x_t, \tag{A.1}$$

where $\nu^0(B,F)$ is the WK filter for $\alpha = 0$, and $h(B,F)$ is the inverse autocorrelation of Lemma 5. From (A.1), removing from $\hat{s}_t^\alpha$ its conditional expectation at time t, it is obtained that

$$r_t^\alpha = \hat{s}_t^\alpha - \hat{s}_{t/t}^\alpha = \sum_{j=1}^{\infty} (\nu_j^0 + h_j\, V_u\, \alpha)\, \varepsilon_t(j), \tag{A.2}$$

where $\varepsilon_t(j)$ is the jth–period–ahead forecast error of x_t, which can be written as

$$\varepsilon_t(j) = a_{t+j} + \sum_{i=1}^{j-1} \psi_i\, a_{t+j-i}, \tag{A.3}$$

where ψ_i is the coefficient of B^i in $\psi(B) = \theta(B)/\phi(B)$. Inserting (A.3) in (A.2), r_t^α can be expressed as

$$r_t^\alpha = \ell_t + m_t\, V_u\, \alpha, \tag{A.4}$$

where $\ell_t = \ell(F)\, a_t$ and $m_t = m(F)\, a_t$, with $\ell_0 = m_0 = 0$ and, for $i \geq 1$,

$$\ell_i = \nu_i^0 + \psi_1\, \nu_{i+1}^0 + \psi_2\, \nu_{i+2}^0 + \dots \tag{A.5.a}$$

$$m_i = h_i + \psi_1\, h_{i+1} + \psi_2\, h_{i+2} + \dots \tag{A.5.b}$$

From (A.4),

$$V(r_t^\alpha) = V(\ell_t) + 2\,\mathrm{Cov}\,(\ell_t, m_t)\, V_u\, \alpha + V(m_t)\, V_u^2\, \alpha^2. \tag{A.6}$$

Since $\ell_t = r_t^\alpha$ when $\alpha = 0$,

$$V(\ell_t) = V(r_t^0). \tag{A.7}$$

From (A.5.b), m_i is seen to be the coefficient of $F^i (i > 0)$ in the polynomial $[h(B,F)\, \psi(B)]$. This polynomial, from the definition of $h(B,F)$ in Lemma 5, is equal to $1/\psi(B)$. Noticing that $m_0 = 0$, we can write

$$1 + m(B) = 1/\psi(B). \tag{A.8}$$

Therefore,

$$V(m_t) = m(B)\, m(F)\, |_0 = (1/\psi(F) - 1)(1/\psi(B) - 1)\, |_0, \tag{A.9}$$

where "$|_0$" denotes "evaluated at $B = F = 0$". Operating in (A.9) yields

$$V(m_t) = h_0 - 1. \tag{A.10}$$

Expression (3.2) implies the identity $\nu^0(B,F)\,\psi(B) = \xi^0(B,F)$, where $\xi^0(B)$ is as in (6.3). Therefore, $\nu^0(B,F) = \xi^0(B,F)(1/\psi(B))$ or, using (A.8),

$$\nu^0(B) = (\ldots + \xi^0_{-1}B + \xi^0_0 + \xi^0_1 F + \ldots)(1 + m_1 B + m_2 B^2 + \ldots).$$

Equating the coefficients for B^0 yields

$$\nu^0_0 = \xi^0_0 + \sum_{i=1}^{\infty} \xi_i\, m_i. \tag{A.11}$$

Expression (A.5.a) implies that ℓ_i is the coefficient of F^i in $\nu^0(B,F)\,\psi(B)$, and hence (A.11) can be rewritten as

$$\mathrm{Cov}\,(\ell_t, m_t) = \nu^0_0 - \xi^0_0, \tag{A.12}$$

since $\sum_{i=1}^{\infty} \xi_i\, m_i = \sum_{i=1}^{\infty} \ell_i\, m_i = \mathrm{Cov}\,(\ell_t, m_t)$. Expressions (A.6), (A.7), (A.10), and (A.12) yield (6.8). ∎

Proof of Lemma 16: From (7.1), $V(d^\alpha_{t|t+k}) = V(e^\alpha_t) + V(r^\alpha_{t|t+k})$. The variance of e^α_t has already been computed in Lemma 5. To obtain $V(r^\alpha_{t|t+k})$ we proceed as follows. Substracting (6.1) from (3.2), it is obtained that

$$r^\alpha_{t|t+k} = \hat{s}^\alpha_t - \hat{s}^\alpha_{t|t+k} = \sum_{i=k+1}^{\infty} \nu^\alpha_i (x_{t+i} - x_{t+i|t+k}), \tag{A.13}$$

where use has been made of the fact that $x_{t+i|t+k} = x_{t+i}$ when $i \le k$. The $(i-k)$–period–ahead forecast error $\varepsilon(i-k) = x_{t+i} - x_{t+i|t+k}$ can be expressed as

$$\varepsilon(i-k) = \sum_{j=0}^{i-k-1} \psi_j\, a_{t+i-j}, \qquad (\psi_0 = 1), \tag{A.14}$$

and inserting (A.14) in (A.13),

$$\begin{aligned} r^\alpha_{t|t+k} = \; & \sum_{i=k+1}^{\infty} \left[(\nu^0_i + \psi_1\, \nu^0_{i+1} + \ldots)\, a_{t+i} + \right. \\ & \left. + \alpha\, V_u (h_i + \psi_1\, h_{i+1} + \ldots)\, a_{t+i} \right], \end{aligned} \tag{A.15}$$

where use has been made of (A.1). Proceeding as in Lemma 7, define

$$\ell_i = \nu^0_i + \psi_1\, \nu^0_{i+1} + \psi_2\, \nu^0_{i+2} + \ldots, \tag{A.16}$$

$$m_i = h_i + \psi_1\, h_{i+1} + \psi_2\, h_{i+2} + \ldots, \tag{A.17}$$

$$\ell_t = \sum_{i=k+1}^{\infty} \ell_i\, a_{t+i}, \tag{A.18}$$

$$m_t = \sum_{i=k+1}^{\infty} m_i\, a_{t+i}, \tag{A.19}$$

Then, we can write

$$V(r^{\alpha}_{t|t+k}) = V(\ell_t) + 2\alpha\, V_u \operatorname{Cov}(\ell_t, m_t) + \alpha^2\, V_u^2\, V(m_t). \tag{A.20}$$

Setting $\alpha = 0$, $V(\ell_t)$ is seen to be the variance of the revision in the canonical specification of the component, that is

$$V(\ell_t) = V(r^{0}_{t|t+k}). \tag{A.21}$$

From (A.19), $V(m_t) = \sum_{i=k+1}^{\infty} m_i^2$, where, according to (A.17), m_i is the coefficient of F^i in the polynomial $h(B,F)\,\psi(B) = 1/\psi(B)$, and hence, $m_i = \lambda_i$, where λ_i was defined in (7.3). Given that $h_0 = \sum_{i=0}^{\infty} \lambda_i^2$, it follows that

$$V(m_t) = h_0 - \delta_k, \tag{A.22}$$

where $\delta_k = \sum_{i=0}^{k} \lambda_i^2$. Finally, $\operatorname{Cov}(\ell_t, m_t) = \sum_{i=k+1}^{\infty} \ell_i\, m_i = \sum_{i=k+1}^{\infty} \lambda_i\, \ell_i$. From (A.16) and (6.10), it is seen that ℓ_i is the coefficient of F^i in $\xi^0(B,F)$, or $\operatorname{Cov}(\ell_t, m_t) = \sum_{i=k+1}^{\infty} \lambda_i\, \xi_i^0$. Using (7.4), it follows that

$$\operatorname{Cov}(\ell_t, m_t) = \nu_0^0 - \zeta_k^0, \tag{A.23}$$

where $\zeta_k^0 = \sum_{i=0}^{k} \xi_i^0\, \lambda_i$. Plugging (A.21), (A.22), and (A.23) in (A.20),

$$V(r^{\alpha}_{t|t+k}) = V(r^{0}_{t|t+k}) + 2(\nu_0^0 - \zeta_k^0)\, V_u\, \alpha + (h_0 - \delta_k)\, V_u^2\, \alpha^2.$$

Adding this variance to $V(e_t^{\alpha})$, given by (4.3), proves the Lemma. ∎

References

BELL, W.R. (1984), "Signal Extraction for Nonstationary Time Series", *Annals of Statistics* 12, 646–664.

BELL, W.R. and HILLMER, S.C. (1984), "Issues Involved with the Seasonal Adjustment of Economic Time Series", *Journal of Business and Economic Statistics* 2, 291–320.

BOX, G.E.P., HILLMER, S.C. and TIAO, G.C. (1978), "Analysis and Modeling of Seasonal Time Series", in Zellner, A. (ed.), *Seasonal Analysis of Economic Time Series*, Washington, D.C.: U.S. Dept. of Commerce — Bureau of the Census, 309–334.

BOX, G.E.P. and JENKINS, G.M. (1970), *Time Series Analysis: Forecasting and Control*, San Francisco: Holden–Day.

BURMAN, J.P. (1980), "Seasonal Adjustment by Signal Extraction", *Journal of the Royal Statistical Society* A, 143, 321–337.

BURRIDGE, P. and WALLIS, K.F. (1985), "Calculating the Variance of Seasonally Adjusted Series", *Journal of the American Statistical Association* 80, 541–552.

CLEVELAND, W.P. (1972), "The Inverse Autocorrelations of a Time Series and their Applications", *Technometrics* 14, 277–298.

CLEVELAND, W.P. and TIAO, G.C. (1976), "Decomposition of Seasonal Time Series: A Model for the X–11 Program", *Journal of the American Statistical Association* 71, 581–587.

ENGLE, R.F. (1978), "Estimating Structural Models of Seasonality", in Zellner, A. (ed.), *Seasonal Analysis of Economic Time Series*, Washington, D.C.: U.S. Dept. of Commerce — Bureau of the Census, 281–297.

FINDLEY, D.F. (1985), "An Analysis and Generalization of M. Watson's Minimax Procedure for Determining the Component Estimates of a Seasonal Time Series", Washington, D.C.: Statistical Research Division, Bureau of the Census.

GARCÍA FERRER, A. and DEL HOYO, J. (1992), "On Trend Extraction Models: Interpretation, Empirical Evidence and Forecasting Performance", *Journal of Forecasting* 11, 645–665.

GRANGER, C.W.J. (1978), "Seasonality: Causation, Interpretation and Implications", in Zellner, A. (ed.), *Seasonal Analysis of Economic Time Series*, Washington, D.C.: U.S. Dept. of Commerce — Bureau of the Census, 33–46.

HARVEY, A.C. (1989), *Forecasting, Structural Time Series Models and the Kalman Filter*, Cambridge: Cambridge University Press.

HILLMER, S.C. (1985), "Measures of Variability for Model–Based Seasonal Adjustment Procedures", *Journal of Business and Economic Statistics* 3, 60–68.

HILLMER, S.C. and TIAO, G.C. (1982), "An ARIMA–Model Based Approach to Seasonal Adjustment", *Journal of the American Statistical Association* 77, 63–70.

HOTTA, L.K. (1989), "Identification of Unobserved Components Models", *Journal of Time Series Analysis* 10, 259–270.

KOHN, R. and ANSLEY, C.F. (1986), "Estimation, Prediction and Interpolation for ARIMA Models with Missing Data", *Journal of the American Statistical Association* 81, 751–761.

MARAVALL, A. (1986), "Revisions in ARIMA Signal Extraction", *Journal of the American Statistical Association* 81, 736–740.

MARAVALL, A. (1988), "A Note on Minimum Mean Squared Error Estimation of Signals with Unit Roots", *Journal of Economic Dynamics and Control* 12, 589–593.

MARAVALL, A. (1994), "Use and Misuse of Unobserved Components in Economic Forecasting", *Journal of Forecasting* 13, 157–178.

MARAVALL, A. and MATHIS, A. (1994), "Encompassing Univariate Models in Multivariate Time Series: A Case Study", *Journal of Econometrics* 61, 197–233.

NERLOVE, M. (1964), "Spectral Analysis of Seasonal Adjustment Procedures", *Econometrica* 32, 241–286.

PIERCE, D.A. (1978), "Seasonal Adjustment when Both Deterministic and Stochastic Seasonality are Present", in Zellner, A. (ed.), *Seasonal Analysis of Economic Time Series*, Washington, D.C.: U.S. Dept. of Commerce — Bureau of the Census, 242–269.

PIERCE, D.A. (1979), "Signal Extraction Error in Nonstationary Time Series", *Annals of Statistics* 7, 1303–1320.

PIERCE, D.A. (1980), "Data Revisions in Moving Average Seasonal Adjustment Procedures", *Journal of Econometrics* 14, 95–114.

STOCK, J.H. and WATSON, M.W. (1988), "Variable Trends in Economic Time Series", *Journal of Economic Perspectives* 2, 147–174.

STOCK, J.H. and WATSON, M.W. (1991), "A Probability Model of the Coincident Economic Indicators", in Lahiri, K. and Moore, G.H. (eds.), *Leading Economic Indicators: New Approaches and Forecasting Records*, Cambridge: Cambridge University Press.

THEIL, H. (1971), *Principles of Econometrics*, New York: J. Wiley and Sons.

WATSON, M.W. (1987), "Uncertainty in Model–Based Seasonal Adjustment Procedures and Construction of Minimax Filters", *Journal of the American Statistical Association* 82, 395–408.

Fig. 1 : SERIES

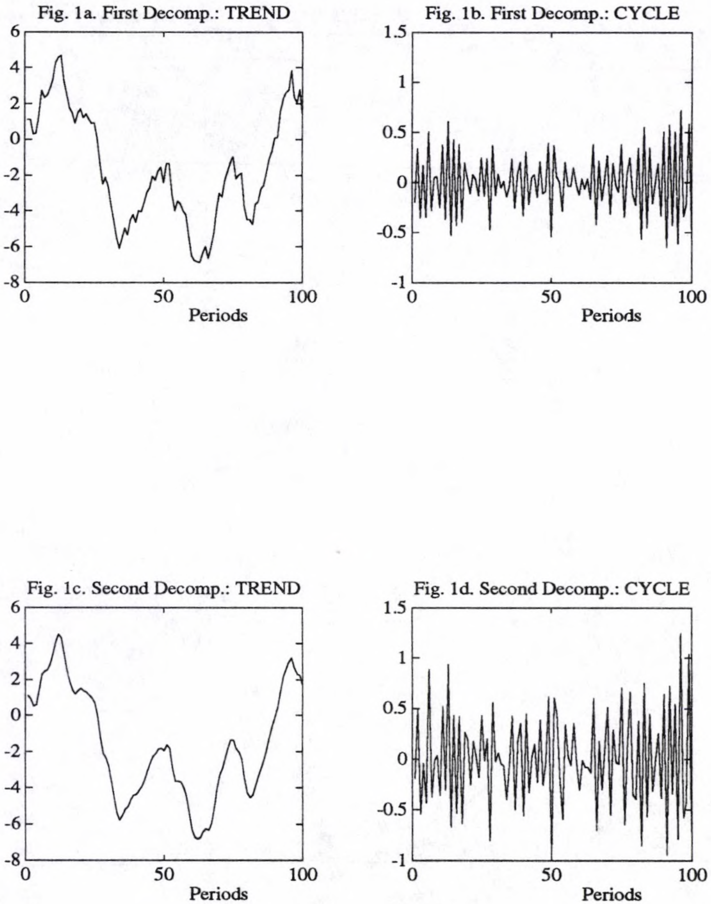


Fig. 1a. First Decomp.: TREND
Fig. 1b. First Decomp.: CYCLE
Fig. 1c. Second Decomp.: TREND
Fig. 1d. Second Decomp.: CYCLE
Periods

Fig. 2 : SPECTRAL DECOMPOSITION

Fig. 2a : SPECTRUM OF SERIES

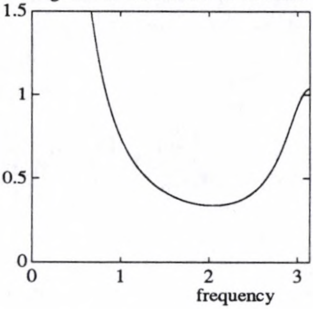


Fig. 2b : FIRST DECOMPOSITION

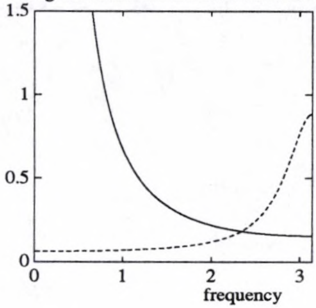


Fig. 2c : ADMISSIBLE TREND SPECTRA

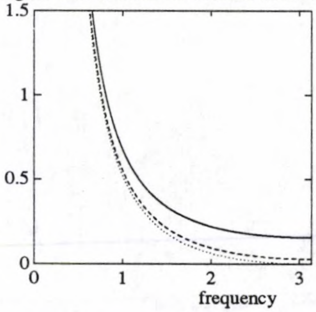


Fig. 2d : ADMISSIBLE CYCLE SPECTRA

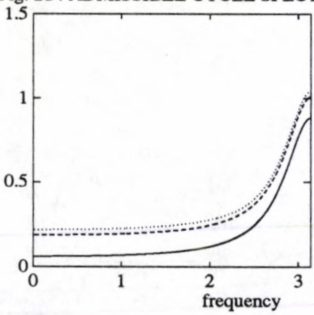
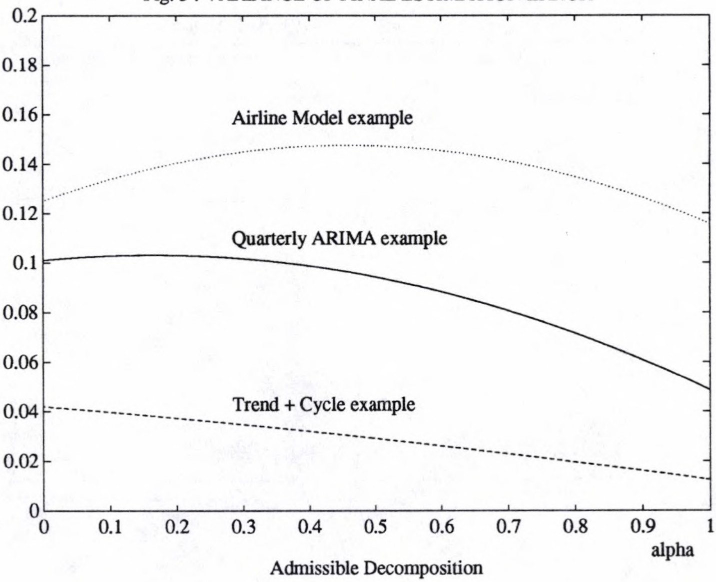
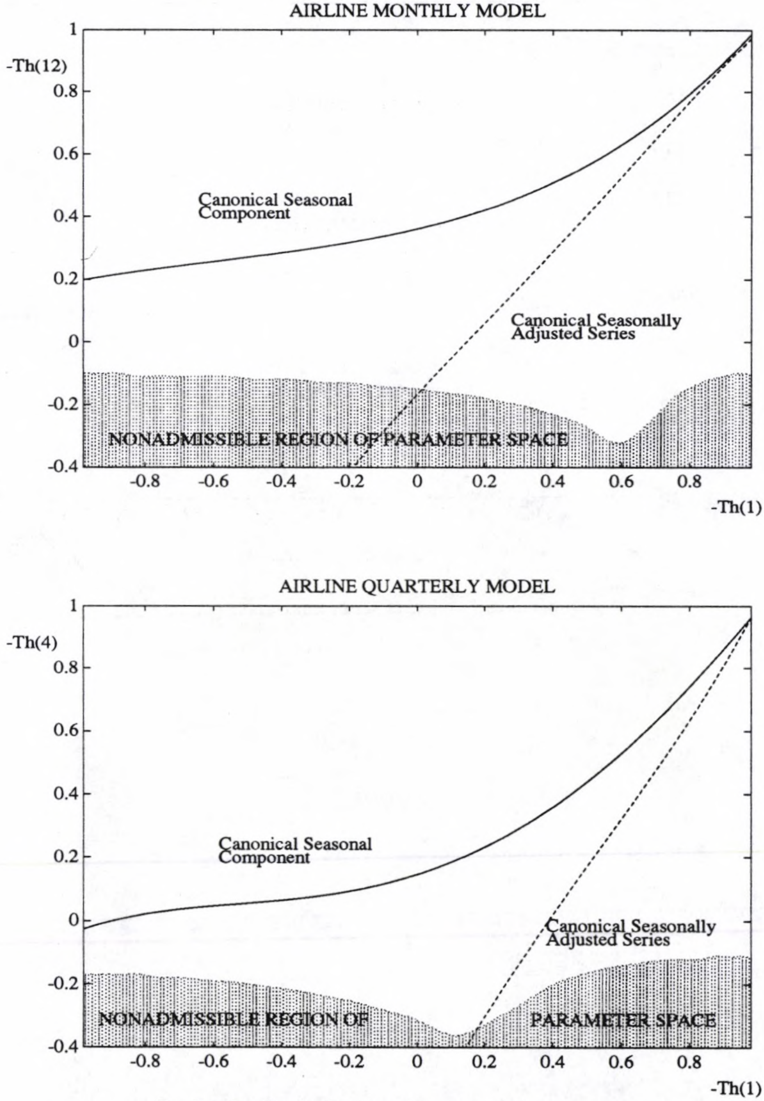


Fig. 3 : VARIANCE OF FINAL ESTIMATION ERROR *



* Expressed as fraction of series innovation variance.

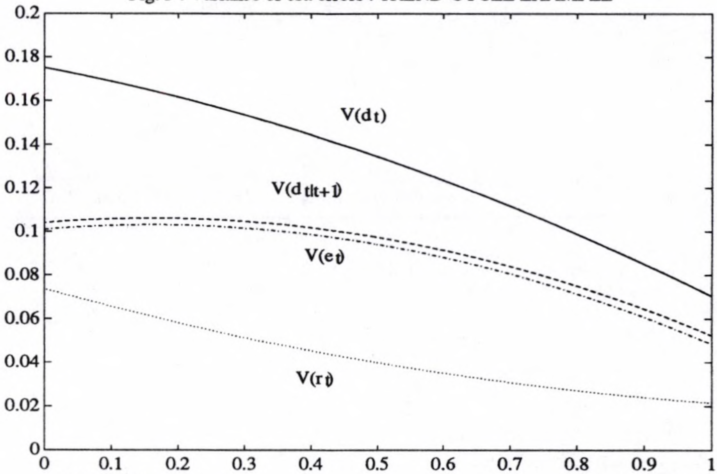
Fig. 4 : CANONICAL SOLUTION THAT MINIMIZES THE ESTIMATION ERROR VARIANCE



— Border line for final estimation error variance

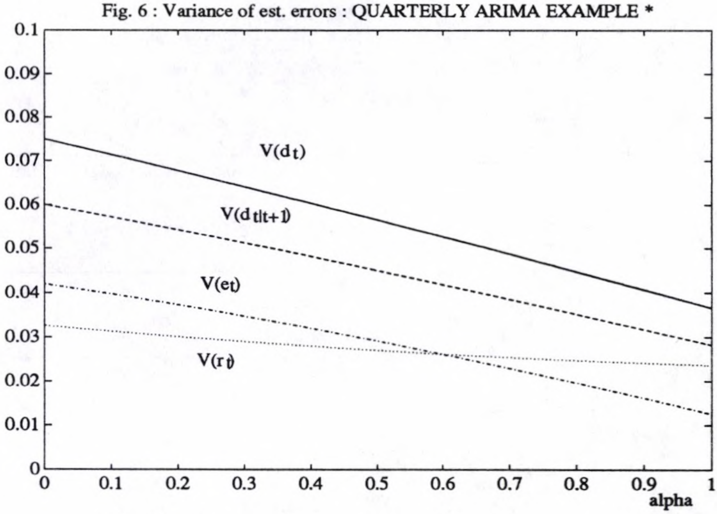
- - Border line for concurrent estimation error variance

Fig. 5 : Variance of est. errors : TREND-CYCLE EXAMPLE *



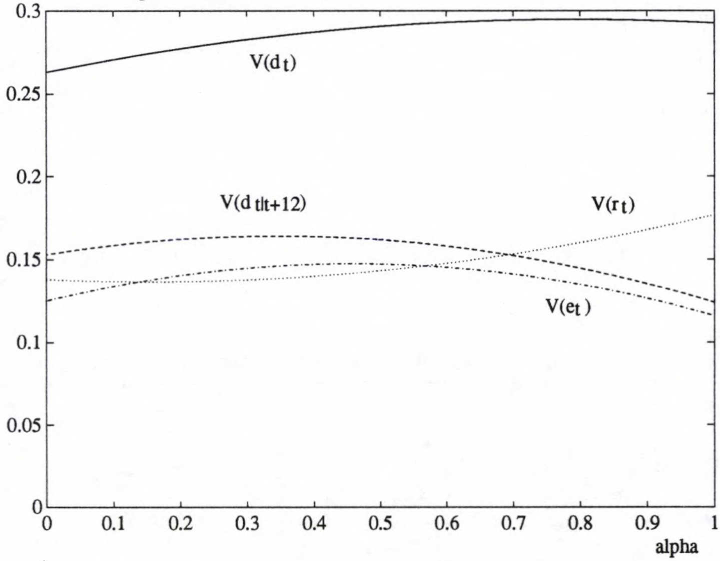
* See Fig.3

Fig. 6 : Variance of est. errors : QUARTERLY ARIMA EXAMPLE *



* See Fig. 3

Fig. 7 : Variance of est. errors : AIRLINE MODEL EXAMPLE *



* See Fig. 3

EUI WORKING PAPERS

EUI Working Papers are published and distributed by the
European University Institute, Florence

Copies can be obtained free of charge
– depending on the availability of stocks – from:

The Publications Officer
European University Institute
Badia Fiesolana
I-50016 San Domenico di Fiesole (FI)
Italy

Please use order form overleaf

Publications of the European University Institute

Department of Economics Working Paper Series

To Department of Economics **WP**
European University Institute
Badia Fiesolana
I-50016 San Domenico di Fiesole (FI)
Italy

From Name . . . .

Address. . . . .

. . . .

. . . .

. . . .

. . . .

(Please print)

❒ Please enter/confirm my name on EUI Economics Dept. Mailing List
❒ Please send me a complete list of EUI Working Papers
❒ Please send me a complete list of EUI book publications
❒ Please send me the EUI brochure Academic Year 1995/96

Please send me the following EUI **ECO** Working Paper(s):

No, Author . . . .
Title: . . . .
No, Author . . . .
Title: . . . .
No, Author . . . .
Title: . . . .
No, Author . . . .
Title: . . . .

Date . . . . Signature . . . .

Working Papers of the Department of Economics Published since 1993

ECO No. 93/1
Carlo GRILLENZONI
Forecasting Unstable and Non-Stationary Time Series

ECO No. 93/2
Carlo GRILLENZONI
Multilinear Models for Nonlinear Time Series

ECO No. 93/3
Ronald M. HARSTAD/Louis PHLIPS
Futures Market Contracting When You Don't Know Who the Optimists Are

ECO No. 93/4
Alan KIRMAN/Louis PHLIPS
Empirical Studies of Product Markets

ECO No. 93/5
Grayham E. MIZON
Empirical Analysis of Time Series: Illustrations with Simulated Data

ECO No. 93/6
Tilman EHRBECK
Optimally Combining Individual Forecasts From Panel Data

ECO NO. 93/7
Víctor GÓMEZ/Agustín MARAVALL
Initializing the Kalman Filter with Incompletely Specified Initial Conditions

ECO No. 93/8
Frederic PALOMINO
Informed Speculation: Small Markets Against Large Markets

ECO NO. 93/9
Stephen MARTIN
Beyond Prices Versus Quantities

ECO No. 93/10
José María LABEAGA/Angel LÓPEZ
A Flexible Demand System and VAT Simulations from Spanish Microdata

ECO No. 93/11
Maozu LU/Grayham E. MIZON
The Encompassing Principle and Specification Tests

ECO No. 93/12
Louis PHLIPS/Peter MØLLGAARD
Oil Stocks as a Squeeze Preventing Mechanism: Is Self-Regulation Possible?

ECO No. 93/13
Pieter HASEKAMP
Disinflation Policy and Credibility: The Role of Conventions

ECO No. 93/14
Louis PHLIPS
Price Leadership and Conscious Parallelism: A Survey

ECO No. 93/15
Agustín MARAVALL
Short-Term Analysis of Macroeconomic Time Series

ECO No. 93/16
Philip Hans FRANSES/Niels HALDRUP
The Effects of Additive Outliers on Tests for Unit Roots and Cointegration

ECO No. 93/17
Fabio CANOVA/Jane MARRINAN
Predicting Excess Returns in Financial Markets

ECO No. 93/18
Iñigo HERGUERA
Exchange Rate Fluctuations, Market Structure and the Pass-through Relationship

ECO No. 93/19
Agustín MARAVALL
Use and Misuse of Unobserved Components in Economic Forecasting

ECO No. 93/20
Torben HOLVAD/Jens Leth HOUGAARD
Measuring Technical Input Efficiency for Similar Production Units: A Survey of the Non-Parametric Approach

ECO No. 93/21
Stephen MARTIN/Louis PHLIPS
Product Differentiation, Market Structure and Exchange Rate Passthrough

ECO No 93/22
F. CANOVA/M. FINN/A. R. PAGAN
Evaluating a Real Business Cycle Model

ECO No 93/23
Fabio CANOVA
Statistical Inference in Calibrated Models

ECO No 93/24
Gilles TEYSSIÈRE
Matching Processes in the Labour Market in Marseilles. An Econometric Study

ECO No 93/25
Fabio CANOVA
Sources and Propagation of International Business Cycles: Common Shocks or Transmission?

ECO No. 93/26
Marco BECHT/Carlos RAMÍREZ
Financial Capitalism in Pre-World War I Germany: The Role of the Universal Banks in the Financing of German Mining Companies 1906-1912

ECO No. 93/27
Isabelle MARET
Two Parametric Models of Demand, Structure of Market Demand from Heterogeneity

ECO No. 93/28
Stephen MARTIN
Vertical Product Differentiation, Intra-industry Trade, and Infant Industry Protection

ECO No. 93/29
J. Humberto LOPEZ
Testing for Unit Roots with the k-th Autocorrelation Coefficient

ECO No. 93/30
Paola VALBONESI
Modelling Interactions Between State and Private Sector in a "Previously" Centrally Planned Economy

ECO No. 93/31
Enrique ALBEROLA ILA/J. Humberto LOPEZ/Vicente ORTS RIOS
An Application of the Kalman Filter to the Spanish Experience in a Target Zone (1989-92)

ECO No. 93/32
Fabio CANOVA/Morten O. RAVN
International Consumption Risk Sharing

ECO No. 93/33
Morten Overgaard RAVN
International Business Cycles: How much can Standard Theory Account for?

ECO No. 93/34
Agustín MARAVALL
Unobserved Components in Economic Time Series *

ECO No. 93/35
Sheila MARNIE/John MICKLEWRIGHT
"Poverty in Pre-Reform Uzbekistan: What do Official Data Really Reveal?"

ECO No. 93/36
Torben HOLVAD/Jens Leth HOUGAARD
Measuring Technical Input Efficiency for Similar Production Units: 80 Danish Hospitals

ECO No. 93/37
Grayham E. MIZON
A Simple Message for Autocorrelation Correctors: DON'T

ECO No. 93/38
Barbara BOEHNLEIN
The Impact of Product Differentiation on Collusive Equilibria and Multimarket Contact

ECO No. 93/39
H. Peter MØLLGAARD
Bargaining and Efficiency in a Speculative Forward Market

***

ECO No. 94/1
Robert WALDMANN
Cooperatives With Privately Optimal Price Indexed Debt Increase Membership When Demand Increases

ECO No. 94/2
Tilman EHRBECK/Robert WALDMANN
Can Forecasters' Motives Explain Rejection of the Rational Expectations Hypothesis?

ECO No. 94/3
Alessandra PELLONI
Public Policy in a Two Sector Model of Endogenous Growth

ECO No. 94/4
David F. HENDRY
On the Interactions of Unit Roots and Exogeneity

ECO No. 94/5
Bernadette GOVAERTS/David F. HENDRY/Jean-François RICHARD
Encompassing in Stationary Linear Dynamic Models

ECO No. 94/6
Luigi ERMINI/Dongkoo CHANG
Testing the Joint Hypothesis of Rationality and Neutrality under Seasonal Cointegration: The Case of Korea

ECO No. 94/7
Gabriele FIORENTINI/Agustín MARAVALL
Unobserved Components in ARCH Models: An Application to Seasonal Adjustment

ECO No. 94/8
Niels HALDRUP/Mark SALMON
Polynomially Cointegrated Systems and their Representations: A Synthesis

ECO No. 94/9
Mariusz TAMBORSKI
Currency Option Pricing with Stochastic Interest Rates and Transaction Costs: A Theoretical Model

ECO No. 94/10
Mariusz TAMBORSKI
Are Standard Deviations Implied in Currency Option Prices Good Predictors of Future Exchange Rate Volatility?

ECO No. 94/11
John MICKLEWRIGHT/Gyula NAGY
How Does the Hungarian Unemployment Insurance System Really Work?

ECO No. 94/12
Frank CRITCHLEY/Paul MARRIOTT/Mark SALMON
An Elementary Account of Amari's Expected Geometry

ECO No. 94/13
Domenico Junior MARCHETTI
Procyclical Productivity, Externalities and Labor Hoarding: A Reexamination of Evidence from U.S. Manufacturing

ECO No. 94/14
Giovanni NERO
A Structural Model of Intra-European Airline Competition

ECO No. 94/15
Stephen MARTIN
Oligopoly Limit Pricing: Strategic Substitutes, Strategic Complements

ECO No. 94/16
Ed HOPKINS
Learning and Evolution in a Heterogeneous Population

ECO No. 94/17
Berthold HERRENDORF
Seigniorage, Optimal Taxation, and Time Consistency: A Review

ECO No. 94/18
Frederic PALOMINO
Noise Trading in Small Markets

ECO No. 94/19
Alexander SCHRADER
Vertical Foreclosure, Tax Spinning and Oil Taxation in Oligopoly

ECO No. 94/20
Andrzej BANIAK/Louis PHLIPS
La Pléiade and Exchange Rate Pass-Through

ECO No. 94/21
Mark SALMON
Bounded Rationality and Learning; Procedural Learning

ECO No. 94/22
Isabelle MARET
Heterogeneity and Dynamics of Temporary Equilibria: Short-Run Versus Long-Run Stability

ECO No. 94/23
Nikolaos GEORGANTZIS
Short-Run and Long-Run Cournot Equilibria in Multiproduct Industries

ECO No. 94/24
Alexander SCHRADER
Vertical Mergers and Market Foreclosure: Comment

ECO No. 94/25
Jeroen HINLOOPEN
Subsidising Cooperative and Non-Cooperative R&D in Duopoly with Spillovers

ECO No. 94/26
Debora DI GIOACCHINO
The Evolution of Cooperation: Robustness to Mistakes and Mutation

ECO No. 94/27
Kristina KOSTIAL
The Role of the Signal-Noise Ratio in Cointegrated Systems

ECO No. 94/28
Agustín MARAVALL/Víctor GÓMEZ
Program SEATS "Signal Extraction in ARIMA Time Series" - Instructions for the User

ECO No. 94/29
Luigi ERMINI
A Discrete-Time Consumption-CAP Model under Durability of Goods, Habit Formation and Temporal Aggregation

ECO No. 94/30
Debora DI GIOACCHINO
Learning to Drink Beer by Mistake

ECO No. 94/31
Víctor GÓMEZ/Agustín MARAVALL
Program TRAMO "Time Series Regression with ARIMA Noise, Missing Observations, and Outliers" - Instructions for the User

ECO No. 94/32
Ákos VALENTINYI
How Financial Development and Inflation may Affect Growth

ECO No. 94/33
Stephen MARTIN
European Community Food Processing Industries

ECO No. 94/34
Agustín MARAVALL/Christophe PLANAS
Estimation Error and the Specification of Unobserved Component Models